A CURL-FREE VECTOR FIELD THAT IS NOT A GRADIENT

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Math 225

Recall our main theorem about vector fields.

Theorem. *Let R be an open region in E^2 and let $\mathbf{F}$ be a C^1 vector field on R. The following statements about $\mathbf{F}$ are equivalent:*

(1) *There is a differentiable function $f\colon R \to \mathbb{R}$ such that $\nabla f = \mathbf{F}$.*
(2) *If C is a piecewise C^1 path in R, then $\int_C \mathbf{F} \cdot d\mathbf{x}$ depends only on the endpoints of C.*
(3) *$\oint_C \mathbf{F} \cdot d\mathbf{x} = 0$ for every piecewise C^1 simple, closed curve in R.*

Furthermore, statements (1)–(3) *imply*

(4) $\operatorname{curl} \mathbf{F} = 0$,

and (4) *implies* (1)–(3) *when R is simply connected (so all four are equivalent when R is simply connected).*

The purpose of this handout is to explore what happens when R is not simply connected, that is, when there exist non-gradient vector fields with zero curl.

An important example

Consider the vector field

$$\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

defined on $R = \mathbb{R}^2 \backslash \{(0,0)\}$, that is, on all of $\mathbb{R}^2$ except the origin. Letting $P = -y/(x^2+y^2)$ and $Q = x/(x^2+y^2)$, it is a simple matter to show that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, and so $\operatorname{curl} \mathbf{F} = 0$. We can show that (1)–(3) are false for $\mathbf{F}$ by finding a simple, closed curve C for which $\oint_C \mathbf{F} \cdot d\mathbf{x} \neq 0$. Let C be the unit circle parameterized counterclockwise by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. We have

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{x} &= \oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \\ &= \int_0^{2\pi} \frac{-\sin t}{1}(-\sin t)\,dt + \frac{\cos t}{1}\cos t\,dt = \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

It follows from the theorem that $\mathbf{F}$ is not the gradient of any function defined on R.

Exercise 1. Show that if C is *any* circle centered at the origin oriented counterclockwise, then $\oint_C \mathbf{F} \cdot d\mathbf{x} = 2\pi$, that is, the radius of the circle does not affect the value of the integral.

Exercise 2. Let D be a disk centered at the origin. Since $\operatorname{curl} \mathbf{F} = 0$, it would be very easy to use Green's Theorem to conclude that $\oint_C \mathbf{F} \cdot d\mathbf{x} = \iint_D \operatorname{curl} \mathbf{F} \, dA = \iint_D 0 \, dA = 0$, where the circle C is the boundary of D, in contrast to the previous exercise. Why is the reasoning here incorrect?

Now suppose that $\tilde{R}$ is a simply connected subset of R. To be concrete, suppose $\tilde{R}$ is all of $\mathbb{R}^2$ except the negative x-axis. Then the theorem implies that $\mathbf{F}$ *is* the gradient of some function $\tilde{f}\colon \tilde{R} \to \mathbb{R}$. Suppose further that $\hat{R}$ is all of $\mathbb{R}^2$ except the positive x-axis. Then $\mathbf{F}$ is the gradient of some function $\hat{f}\colon \hat{R} \to \mathbb{R}$. Then the functions $\tilde{f}$ and $\hat{f}$ have the same gradient, namely $\mathbf{F}$, on their common domain $\tilde{R} \cap \hat{R}$. You might guess that if two functions have the same gradient, then they must differ by a constant. That's true, but only if you restrict yourself to a connected set. In this case the common domain $\tilde{R} \cap \hat{R}$ consists of *two* disjoint connected subsets, namely the upper and lower half planes ($y > 0$ on one and $y < 0$ on the other). Evidently the two functions differ by different constants on these two half planes, so you can't "fix" the situation by adding a constant to one of them.

What can these functions be? Given a vector field with zero curl, you have a method for finding a function whose gradient is the vector field. It's a bit messy to apply with this vector field, so I'll just tell you what the answer is: it's θ, the angle of polar coordinates, plus an arbitrary constant, of course.

Exercise 3. Verify this by implicit differentiation. To do this, first show that $x \sin\theta = y \cos\theta$ (can you do this without dividing by anything that might be zero?). Then apply $\frac{\partial}{\partial x}$ to both sides of this, treating y as a constant and θ as a function of x and y.

Now you may be thinking that there is a contradiction here: first we decided that $\mathbf{F}$ is not the gradient of any function on R, and then we showed that it is the gradient of θ! The resolution of the apparent contradiction is that θ is not a well-defined function on R, even though we often treat it as such. If you start with $\theta = 0$ on the positive x-axis and then go counterclockwise around the origin, θ increases. When you get close to the positive x-axis in the fourth quadrant, θ is near 2π. There is no way to make it a continuous, let alone differentiable, function on R. What about on $\tilde{R}$ and $\hat{R}$? On $\tilde{R}$ you can take θ to have values between $-\pi$ and π. On $\hat{R}$ you can take θ to have values between 0 and 2π. More generally, if $\tilde{R}$ is an arbitrary simply connected subset of R, then $\mathbf{F}$ is the gradient of some function $\tilde{f}\colon \tilde{R} \to \mathbb{R}$. By adding a constant, you can get $\tilde{f} + C$ to agree with some suitable definition of θ on $\tilde{R}$, but it may not agree with any of the "usual" definitions of θ on all of R.

Exercise 4. Suppose that $\tilde{R}$ is a connected, narrow spiral-shaped subregion of R that contains $(1, 0)$ and $(5, 0)$, but that to get from $(1, 0)$ to $(5, 0)$, staying in the region, you have to cross the negative x-axis. If $\tilde{f}(1, 0) = 0$, then $\tilde{f}$ agrees with the usual definition of θ near $(1, 0)$. What is $\tilde{f}(5, 0)$ in this case? Notice that $\tilde{f}$ is well-defined, but that it takes on a range of values larger than 2π!

Exercise 5. Show that the following vector field, also with domain R, is the gradient of a function defined on all of R:

$$\frac{x}{x^2+y^2}\mathbf{i}+\frac{y}{x^2+y^2}\mathbf{j}$$

The general case on $R=\mathbb{R}^2\setminus\{(0,0)\}$

Now suppose that $\mathbf{G}$ is an arbitrary vector field defined on $R=\mathbb{R}^2\setminus\{(0,0)\}$, that $\operatorname{curl}\mathbf{G}=0$, and that $\mathbf{G}$ is not the gradient of any function. By the theorem, there must be a simple, closed curve C such that $\oint_C \mathbf{G}\cdot d\mathbf{x}\neq 0$. Let's find one.

Suppose that C is a simple, closed curve in R. If C does not go around the origin, then C is contained in some simply connected subregion $\tilde{R}$ of R. Then the theorem implies that $\oint_C \mathbf{G}\cdot d\mathbf{x}=0$. (You can also conclude this by applying Green's Theorem to the region bounded by C.) Thus, if C is a simple, closed curve for which $\oint_C \mathbf{G}\cdot d\mathbf{x}\neq 0$, then C must go around the origin.

Suppose that C_1 and C_2 are two simple, closed curves in R that go around the origin counterclockwise. Furthermore, assume that they don't intersect and that C_1 is the outer curve. (Draw a picture!) Let Ω be the annular region between C_1 and C_2. Then $\partial\Omega$ is $C_1\cup C_2$, which is given its correct orientation by having C_1 go counterclockwise and C_2 go clockwise. By Green's Theorem we have

$$0=\iint_\Omega \operatorname{curl} G\,dA=\oint_{\partial\Omega}\mathbf{G}\cdot d\mathbf{x}=\oint_{C_1}\mathbf{G}\cdot d\mathbf{x}-\oint_{C_2}\mathbf{G}\cdot d\mathbf{x},$$

and so

$$\oint_{C_1}\mathbf{G}\cdot d\mathbf{x}=\oint_{C_2}\mathbf{G}\cdot d\mathbf{x}.$$

Exercise 6. Show that this holds even if C_1 and C_2 intersect.

Thus every simple, closed curve C that goes around the origin counterclockwise gives the same non-zero value for the integral $\oint_C \mathbf{G}\cdot d\mathbf{x}$. (This should remind you of Exercise 1.) This number is a property of the vector field $\mathbf{G}$. Denote this value by c.

Define a new vector field $\widetilde{\mathbf{G}}$ by modifying $\mathbf{G}$:

$$\widetilde{\mathbf{G}}=\mathbf{G}-\frac{c}{2\pi}\mathbf{F},$$

where $\mathbf{F}$ is the vector field in the previous section.

I claim that $\widetilde{\mathbf{G}}$ is the gradient of some function on R. To see this, suppose C is a simple, closed curve in R. If C doesn't go around the origin, then, as observed above, $\oint_C \mathbf{G}\cdot d\mathbf{x}=0$. For the same reason, $\oint_C \mathbf{F}\cdot d\mathbf{x}=0$ (in fact $\mathbf{F}$ is a special case of $\mathbf{G}$). Thus $\oint_C \widetilde{\mathbf{G}}\cdot d\mathbf{x}=0$.

Now suppose C goes around the origin counterclockwise. Then

$$\oint_C \widetilde{\mathbf{G}}\cdot d\mathbf{x}=\oint_C \mathbf{G}\cdot d\mathbf{x}-\frac{c}{2\pi}\oint_C \mathbf{F}\cdot d\mathbf{x}=c-\frac{c}{2\pi}(2\pi)=0.$$

Since $\oint_C \mathbf{G} \cdot d\mathbf{x} = 0$ for every simple, closed curve C in R, the theorem implies that $\widetilde{\mathbf{G}}$ is the gradient of some function $\tilde{g} \colon R \to \mathbb{R}$. Recall that $\mathbf{F}$ is almost the gradient of θ, except that θ isn't a well-defined function on R. Similarly, $\mathbf{G}$ is almost the gradient of $\tilde{g} + c\theta$. On a simply connected subregion of R, $\mathbf{G}$ will be the gradient of $\tilde{g} + c\theta$, where θ is some extension of the usual θ to the subregion.

Note that this says that every vector field $\mathbf{G}$ on R that is curl-free and not a gradient can be made into a gradient simply by adding a constant multiple of the vector field $\mathbf{F}$ that is the main example,

$$\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

Thus the vector field $\mathbf{F}$ is, in some sense, the *only* example, at least on R.

The case of more holes

Now suppose that $R \subset \mathbb{R}^2$ is a region with finitely many holes. A hole can be more than just a point; can be the result of removing a simply connected set. In general, $R = R_0 \setminus (H_1 \cup \cdots \cup \bar{H}_n)$ in which R_0 is a simply connected region, each H_i is a non-empty, closed set that is either a point or the closure of a simply connected region, and the sets $H_1, \ldots, H_n$ are disjoint subsets of R_0.

For each index i, let $P_i(x_i, y_i)$ be a point in H_i, and consider the vector field

$$\mathbf{F}_i = \frac{-(y-y_i)}{(x-x_i)^2+(y-y_i)^2}\mathbf{i} + \frac{x-x_i}{(x-x_i)^2+(y-y_i)^2}\mathbf{j}.$$

This is similar to the main example $\mathbf{F}$, but it is centered at P_i instead of the origin. If $\mathbf{r}$ denotes the position vector of a point X from the origin and r denotes the distance from X to the origin, note that $\mathbf{F}$ can be written as $\mathbf{F} = \mathbf{r}_\perp / r^2$. In the same way, let $\mathbf{r}_i = X - P_i$ be the position vector of X relative to P_i, and let $r = \|X - P_i\| = \|\mathbf{r}_i\|$ be the distance from P_i to X. Then we have $\mathbf{F}_i = (\mathbf{r}_i)_\perp / r_i^2$.

If C is a simple, closed curve in R oriented counterclockwise, then $\oint_C \mathbf{F}_i \cdot d\mathbf{x}$ is equal to 2π or 0 depending on whether C goes around P_i or not. More generally, suppose that C is piecewise C^1 loop in R. It can be shown that $\frac{1}{2\pi} \oint_C \mathbf{F}_i \cdot d\mathbf{x}$ is an integer. It is called the winding number of C around P_i because it counts how many times C goes around P_i. Since C stays in R, this integer also counts the number of times C goes around H_i. The collection of integers, $\frac{1}{2\pi} \oint_C \mathbf{F}_i \cdot d\mathbf{x}$ for $1 \le i \le n$, gives considerable information about how the curve C makes its way around all of the holes in the region.

Exercise 7. Let $P_1(0,0)$, $P_2(1,0)$, $P_3(0,1)$, and let $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ be the corresponding vector fields. Draw a closed curve C such that

$$\frac{1}{2\pi}\oint_C \mathbf{F}_1 \cdot d\mathbf{x} = 2, \qquad \frac{1}{2\pi}\oint_C \mathbf{F}_2 \cdot d\mathbf{x} = 1, \qquad \text{and} \qquad \frac{1}{2\pi}\oint_C \mathbf{F}_3 \cdot d\mathbf{x} = -1.$$

Exercise 8. Suppose P_1 is a point and $\mathbf{F}_1$ is the corresponding vector field. If C is a curve such that $\oint_C \mathbf{F}_1 \cdot d\mathbf{x} = 0$, it can be shown that C can be morphed to a point (a curve that doesn't go anywhere) without passing through P_1 during the morphing process. (This is a substantial theorem. It says that no matter how crazy the curve C is, if $\oint_C \mathbf{F}_1 \cdot d\mathbf{x} = 0$, then C can be "unwound" from around P_1.) Let $P_1(0,0)$ and $P_2(1,0)$, and let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the corresponding vector fields. Draw a closed curve C such that

$$\oint_C \mathbf{F}_1 \cdot d\mathbf{x} = \oint_C \mathbf{F}_2 \cdot d\mathbf{x} = 0$$

for which it is intuitively clear that C *cannot* be morphed to a point without passing through at least one of the points.

Suppose $\mathbf{G}$ is a vector field on R with $\operatorname{curl} \mathbf{G} = 0$. For each i, let C_i be a simple, closed curve in R, oriented counterclockwise, that goes around the hole H_i, but none of the other holes. Let $c_i = \oint_{C_i} \mathbf{F}_i \cdot d\mathbf{x}$, and define a new vector field by

$$\widetilde{\mathbf{G}} = \mathbf{G} - \frac{1}{2\pi} \sum_{i=1}^{n} c_i \mathbf{F}_i.$$

Then it can be shown that $\widetilde{\mathbf{G}}$ is the gradient of some function on R. This implies that if $\operatorname{curl} \mathbf{G} = 0$, then the only difference between $\mathbf{G}$ and a gradient is some linear combination of the vector fields $\mathbf{F}_1, \ldots, \mathbf{F}_n$.

Exercise 9. Let $\widetilde{\mathbf{G}}$ be the vector field in the paragraph above. Prove that $\widetilde{\mathbf{G}}$ is the gradient of some function on R. Do this by supposing that C is a piecewise C^1, simple, closed curve in R, and giving a careful explanation of why $\oint_C \widetilde{\mathbf{G}} \cdot d\mathbf{x} = 0$.