INFORMAL DERIVATIONS OF ARC-LENGTH FORMULAS

In class we saw that if $\gamma\colon [a,b] \to E^n$ is continuously differentiable, then its length is given by

$$\ell(\gamma) = \int_a^b \|\gamma'(t)\|\, dt. \tag{1}$$

This formula can be found informally by computing with infinitesimals. For simplicity we work with coordinates in E^2, assuming $\gamma(t) = (x(t), y(t))$. If ds is length of an infinitesimal part of the curve, the infinitesimal Pythagorean theorem gives us

$$ds^2 = dx^2 + dy^2.$$

(Another way to think of this is that ds is the length of the infinitesimal displacement vector $dx\,\mathbf{i} + dy\,\mathbf{j}$.) The length of the curve is $\ell(\gamma) = \int_C ds$, where $\int_C$ means that we integrate along the curve. We have

$$\ell(\gamma) = \int_C ds = \int_C \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt, \tag{2}$$

which is equal to the formula for $\ell(\gamma)$ in (1) since $\gamma'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$.

Note that if the curve is the graph of $y = f(x)$, $a \le x \le b$, where f is continuously differentiable, a similar computation yields

$$\ell(\gamma) = \int_C ds = \int_C \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + f'(t)^2}\, dt,$$

which is the formula you learned for the length of a graph in Math 112. Note that this can be obtained from (2) simply by letting $t = x$ (x is the parameter!).

With some hindsight, we can derive (1) directly without coordinates. If γ represents a point on the curve, then $d\gamma$ represents an infinitesimal displacement vector from one point on the curve to another point infinitesimally far away. The infinitesimal distance traveled is $ds = \|d\gamma\|$. (This looks a bit strange. In E^2 we would have $\gamma = (x, y)$, $d\gamma = dx\,\mathbf{i} + dy\,\mathbf{j}$ and $ds = \|d\gamma\| = \sqrt{dx^2 + dy^2}$.) Then

$$\ell(\gamma) = \int_C ds = \int_C \|d\gamma\| = \int_a^b \left\|\frac{d\gamma}{dt}\right\| dt = \int_a^b \|\gamma'(t)\|\, dt.$$

Polar Coordinates. Sometimes a curve in E^2 is expressed in polar coordinates, either with r as a function of θ, or parametrically with r and θ as functions of some other variable. The formulas for arc-length in terms of r and θ can be derived by using the formulas that relate polar and Cartesian coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Taking differentials of these yields

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta, \qquad dy = \sin\theta\, dr + r\cos\theta\, d\theta.$$

If you don't remember how to take differentials from Math 112 (MVC Section 3) another way to do this is to think of r and θ as functions of t (which we are implicitly doing anyway, since we are moving along a curve) and differentiate with respect to t. We get

$$\frac{dx}{dt} = \cos\theta\,\frac{dr}{dt} - r\sin\theta\,\frac{d\theta}{dt}, \qquad \frac{dy}{dt} = \sin\theta\,\frac{dr}{dt} + r\cos\theta\,\frac{d\theta}{dt}.$$

Then multiply by dt.)

Now do the algebra. Square these expressions for dx and dy and add the results. We get

$$ds^2 = dx^2 + dy^2 = \cdots = dr^2 + r^2\, d\theta^2.$$

(You should fill in the details.) The length of the curve is then

$$\ell(\gamma) = \int_C ds = \int_C \sqrt{dr^2 + r^2\, d\theta^2}.$$

If r and θ are functions of t, $a \le t \le b$, we get

$$\ell(\gamma) = \int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}\, dt = \int_a^b \sqrt{r'(t)^2 + r^2\theta'(t)^2}\, dt.$$

If r is a function of θ, $\alpha \le \theta \le \beta$, we get

$$\ell(\gamma) = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta = \int_\alpha^\beta \sqrt{r'(\theta)^2 + r(\theta)^2}\, d\theta.$$

There are similar formulas for curves in E^3 in terms of cylindrical and spherical coordinates.

Historical Note. These informal derivations with infinitesimals are fairly close to how these formulas were originally discovered. They are not rigorous, however, because we haven't given a rigorous definition of infinitesimals. They are intuitive, though, which is why it's worth learning and using them.

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