Math 223 **Final Exam** **Name:**

4 May 2011 200 Points

"Show enough work to justify your answers."

No calculators.

READ CAREFULLY! This exam has **three** parts. You have choices in each part.

Part I. Definitions (50 points). Do **exactly ten** problems in this part. If you do more than ten, I will grade the first ten and ignore the rest. *If you begin one and decide you do not want it graded, cross your work out.*

Define the following terms. Some of the terms may be used in the definitions of other terms. In some cases when a definition involves a formula, the formula by itself is almost never the complete definition. (5 points each)

1. The rank of matrix A is ...

2. Let S be a subset of a vector space V. S is closed under addition if ...

3. A linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is ...

4. The span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is ...

5. $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent if ...

6. A basis for a vector space V is ...

7. The dimension of a vector space V is ...

8. Two vectors $\mathbf{v}$ and $\mathbf{w}$ in an inner product space are orthogonal (perpendicular) if ...

9. A square matrix A is skew-symmetric (or anti-symmetric) if ...

10. If V is a subspace of $\mathbb{R}^n$, the orthogonal complement $V^\perp$ of V is ...

11. If V and W are vector spaces, a function $T : V \to W$ is linear if ...

12. Define the column space and row space of a matrix A.

13. If A is a matrix, the null space of A is ...

14. $\mathcal{C}^2([0, 3])$ is ...

15. Define eigenvector and eigenvalue of a matrix A.

Part II. Computational Problems (75 points). Do **at least five** problems in this part. If you do more than five, you will get credit for the best five. (15 points each)

1. Solve the following linear system. Write the results in parametric form.

$$\begin{array}{rcrcrcrcr} 2w & + & 3x & - & 4y & + & 2z & = & 4 \\ w & & & + & y & + & z & = & 5 \\ -3w & + & 2x & - & 2y & & & = & -2 \end{array}$$

2. Compute the inverse of the following matrix or show that the inverse does not exist.

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}$$

3. Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$$

4. Given that the eigenvalues of A are 3 and -2, find a basis for each eigenspace of A.

$$A = \begin{pmatrix} -2 & 10 & 5 \\ 10 & -17 & -10 \\ -20 & 40 & 23 \end{pmatrix}$$

5. Let A be the matrix below.

   (a) Find a basis of the column space of A consisting of columns of A.

   (b) Is the column space all of $\mathbb{R}^3$? Why or why not?

$$A = \begin{pmatrix} -1 & 0 & -1 & -2 & 3 \\ 3 & -1 & 1 & 7 & -1 \\ 3 & 2 & 7 & 4 & 3 \end{pmatrix}$$

6. Let $\mathbf{v} = (1, 2, -1)$. Define $L : \mathbb{R}^3 \to \mathbb{R}^3$ by $L(\mathbf{x}) = \text{Proj}_{\mathbf{v}}\,\mathbf{x}$. Find the matrix for L.

7. Compute the following determinant.

$$\det \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 3 & 0 & 2 \\ 2 & 1 & -3 & 6 \\ 3 & 3 & 4 & 2 \end{pmatrix}$$

8. Find the dimension of the span of $\mathbf{v}_1 = (2, -1, 1, 3)$, $\mathbf{v}_2 = (1, 1, 3, 2)$, and $\mathbf{v}_3 = (1, -5, -7, 0)$. Explain.

Part III. Theoretical Problems (75 points). Do **at least five** problems in this part. If you do more than five, you will get credit for the best five. (15 points each)

1. Consider the following system, in which k is a constant.

$$\begin{array}{rcrcl} x & + & (2-k)y & = & 1 \\ (3+k)x & + & 4y & = & 4 \end{array}$$

   Determine the following with reasoning.

   (a) For which values of k does the system have exactly one solution?

   (b) For which values of k does the system have no solutions?

   (c) For which values of k does the system have infinitely many solutions?

2. Prove that $f(x) = \cos x$, $g(x) = e^x$, and $h(x) = e^{-x}$ are independent.

3. Find an orthogonal basis for $\mathcal{P}_1$ relative to the inner product $\langle f, g\rangle = \int_0^2 f(x)g(x)\,dx$.

4. Let V be the set of all functions f in $\mathcal{C}^0(\mathbb{R})$ such that $\int_{-1}^1 xf(x)\,dx = 0$.

   (a) Prove that V is a subspace of $\mathcal{C}^0(\mathbb{R})$. (9 points)

   (b) Find, with verification, two non-zero functions, one of which is in V and one of which is not in V. (6 points)

5. Let $\mathcal{C}^\infty(\mathbb{R})$ be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$ that can be differentiated infinitely many times. Define $T : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ by $T(f) = f''$.

   (a) Show that $f(x) = e^{-2x}$ is an eigenfunction of T, and determine its eigenvalue. (2 points)

   (b) Give another eigenfunction g of T with the same eigenvalue as f so that f and g are independent. Verify. (3 points)

   (c) Give two independent eigenfunctions of T with eigenvalue -5. Verify. (5 points)

   (d) Give two independent eigenfunctions of T with eigenvalue 0. Verify. (5 points)

6. Suppose A is a 3×3 matrix and that $\det(A) = 5$. What is $\det(2A)$? Explain.

7. Define $T : \mathcal{P}_2 \to \mathcal{P}_2$ by $T(p(x)) = p(x + 2)$. For example, $T(x^2) = (x + 2)^2$. This is linear (you do not need to prove it.) Find the matrix of T relative to the basis $\mathcal{B} = \{1, x, x^2\}$.

8. Let $\mathbf{v}_1 = (1, 3)$ and $\mathbf{v}_2 = (1, 2)$. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and that $T(\mathbf{v}_1) = 2\mathbf{v}_1$ and $T(\mathbf{v}_2) = -\mathbf{v}_2$. Let $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2]$. Find the matrix $[T]_\mathcal{B}$ of T relative to the basis $\mathcal{B}$ and also the matrix $[T]_\mathcal{E} = [T]_{stand}$ of T relative to the standard basis.

Have a Good Summer!

Partial answers and hints.

Part II

1. $(8/5, 24/5, 17/5, 0) + t(2, 6, 3, -5)$. Remember you can check your answer to some extent by plugging in: $(8/5, 24/5, 175, 0)$ should satisfy the given system; $(2, 6, 3, -5)$ should satisfy the associated homogeneous system. What this won't catch is not having enough free variables.

2. Check your answer by multiplication.

3. 6 and 1

4. A basis for $E_3(A)$ consists of $(2, 1, 0)$ and $(1, 0, 1)$. A basis for $E_{-2}(A)$ consists of $(1, -2, 4)$.

5. A basis consists of the first, second, and last columns. These do form a basis for $\mathbb{R}^3$ because they are three independent vectors in $\mathbb{R}^3$.

6. $\frac{1}{6}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

7. -2. By doing suitable row and column operations, this can be reduced to a single 2×2 determinant.

8. 2

Part III

1. When $k = 1$ there are infinitely many solutions. When $k = -2$ there are no solutions. For all other values of k there is exactly one solution.

3. One possible orthogonal basis consists of 1 and $x - 1$.

5. (b) e^{2x} (c) $\sin(\sqrt{5}x)$ and $\cos(\sqrt{5}x)$ (d) x and 1
In each part, since there are only two functions, independence is verified by observing that neither function is a constant multiple of the other.

6. 40

7. The operation is to replace every instance of x with $x + 2$. The matrix is $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

8. $[T]_{\mathcal{B}} = \begin{pmatrix} -7 & 3 \\ -18 & 8 \end{pmatrix}$