**Math 223** **Exam 3** **Name:**

25 April 2011 100 Points

You may use a calculator to do arithmetic, but not to solve systems or do matrix algebra. Exact answers are expected.

*"Show enough work to justify your answers."*

1. Do **one** of the following. Note that (a) has two parts. (20 points)

   (a) Define $p_1, p_2 \in \mathcal{P}_1$ by $p_1(t) = t - 1$ and $p_2(t) = 3t - 1$. These form a basis for $\mathcal{P}_1$ (you do not need to prove this).

   - Show that $p_1$ and $p_2$ are orthogonal in the standard inner product of $\mathcal{C}^0([0,1])$.
   - Find the projection of $f(x) = t^2$ into $\mathcal{P}_1$ relative to this inner product.

   (b) Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = (1,0,1)$ and $\mathbf{v}_2 = (1,1,0)$. Find the projection into $V$ of $\mathbf{b} = (1,1,1)$. Warning: $\mathbf{v}_1$ and $\mathbf{v}_2$ are not orthogonal.

2. Do **one** of the following. (20 points)

   (a) Let $f_1(t) = \sin 2t$, $f_2(t) = \cos 2t$, $f_3(t) = e^t$, and $f_4(t) = e^{2t}$. These form a basis for $V = \text{span}\{f_1, f_2, f_3, f_4\}$. Define $T : V \to V$ by $T(f) = f' - f$. Find the matrix for $T$ relative to the basis $\mathcal{B} = [f_1, f_2, f_3, f_4]$.

   (b) Let $\mathcal{S}$ be the subspace of symmetric $2\times 2$ matrices. A basis for $\mathcal{S}$ is $\mathcal{B} = [F_1, F_2, F_3]$, where

   $$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and define $T : \mathcal{S} \to \mathcal{S}$ by $T(X) = AX + XA^T$. Find the matrix of $T$ relative to the basis $\mathcal{B}$.

3. Do **two** of the following. Note that (b) and (c) have two parts. (20 points each)

   (a) Suppose that $\mathbf{v}, \mathbf{w}$ are eigenvectors of the matrix $A$ with eigenvalues of 3 and 5, respectively. Prove that $\mathbf{v}$ and $\mathbf{w}$ are independent.

   (b)
   - Prove that $f_1(t) = \sin 2t$, $f_2(t) = \cos 2t$, $f_3(t) = e^t$, and $f_4(t) = e^{2t}$ are independent.
   - Use a familiar trig identity to prove that $g_1(t) = \sin^2 t$, $g_2(t) = \cos^2 t$, and $g_3(t) = 1$ are dependent.

   (c) Let $\mathbf{v}_1 = (1,0,1)$, $\mathbf{v}_2 = (1,1,0)$, and $\mathbf{v}_3 = (-1,1,1)$. Note that $\mathbf{v}_3$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be projection onto $V$.

   - Explain why the matrix for $T$ relative to $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ is $[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. (5 points)
   - Find the matrix for $T$ relative to the standard basis. (15 points) Note that $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 1 \end{pmatrix} = 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

4. Find the volume of the parallelepiped determined by $\mathbf{a} = (2, 3, -2)$, $\mathbf{b} = (1, -3, 2)$, and $\mathbf{c} = (1, 1, 0)$. You may use any method you like to compute the determinant. (20 points)

Partial answers and hints.

1. (a) Note: The standard basis of $\mathcal{P}_1$ is not orthogonal for most inner products. The projection of $t^2$ is $t - 1/6$.

   (b) $\frac{2}{3}(2, 1, 1)$

2. (a) $[T]_{\mathcal{B}} = \begin{pmatrix} -1 & -2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (b) The answer is a $3 \times 3$ matrix.

3. (b) To prove that the coefficients are zero requires four independent equations.

   (c) The last sentence tells you the inverse of the new basis matrix. The matrix is $[T]_{\mathcal{E}} = \frac{1}{3} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

4. Volume isn't negative!