Math 223 **Exam 2** **Name:**

30 March 2011 100 Points

You may use a calculator to do arithmetic, but exact answers are expected.

"Show enough work to justify your answers."

1. Consider the subspaces V and W, and the matrix A and its reduced form R. Note how the spanning sets for V and W are related to A. (15 points)

$$V = \text{span}\{(-4, 11, -19, -6, 24),\ (1, -2, 4, 2, -7),\ (-2, 6, -10, -3, 12)\}$$

$$W = \text{span}\{(-4, 1, -2),\ (11, -2, 6),\ (-19, 4, -10),\ (-6, 2, -3),\ (24, -7, 12)\}$$

$$A = \begin{pmatrix} -4 & 11 & -19 & -6 & 24 \\ 1 & -2 & 4 & 2 & -7 \\ -2 & 6 & -10 & -3 & 12 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

(a) Give a basis for V. (4 points)

(b) Give a basis for W. (4 points)

(c) Give a basis for $N(A)$. (4 points)

(d) Give the dimensions. (3 points)

$\dim V =$ $\dim W =$ $\dim N(A) =$

2. Definitions. Do **two** of the following. If you do more than two, you will get credit for the best two. (10 points)

(a) Define what it means for a square matrix M to be skew-symmetric (or anti-symmetric).

(b) If V is a subspace of $\mathbf{R}^n$, define $V^\perp$.

(c) Define what it means for a collection $\mathbf{v}_1, \ldots \mathbf{v}_k$ in $\mathbf{R}^n$ to be linearly independent.

3. Multiple Choice. For each of the following, state whether it *must* be true, *might* be true, or *can't* be true. (Circle one for each question.) (15 points)

(a) If $\mathbf{v}_1, \ldots \mathbf{v}_5$ are in $\mathbf{R}^8$, then they are independent. must might can't

(b) If $\mathbf{v}_1, \ldots \mathbf{v}_5$ are in $\mathbf{R}^8$, then they span $\mathbf{R}^8$. must might can't

(c) If $\mathbf{v}_1, \ldots \mathbf{v}_8$ are in $\mathbf{R}^5$, then they are independent. must might can't

(d) If $\mathbf{v}_1, \ldots \mathbf{v}_8$ are in $\mathbf{R}^5$, then they span $\mathbf{R}^5$. must might can't

(e) If $\mathbf{v}_1, \ldots \mathbf{v}_8$ are in $\mathbf{R}^8$ and they are independent, then they span $\mathbf{R}^8$.
must might can't

4. Do any **two** of the following. If you do more than two, you will get credit for the best two. Use the back of the page if you need more room. (30 points)

   (a) Use the definition of linear independence to determine if $\mathbf{v}_1 = (3, 2, -1, 1)$, $\mathbf{v}_2 = (2, 2, 1, 1)$, and $\mathbf{v}_3 = (1, 2, 3, 1)$ are linearly independent.

   (b) Let $A = \left(\begin{smallmatrix} 2 & 4 \\ 1 & 3 \end{smallmatrix}\right)$. Reduce A to the identity, and use your steps to find a sequence of elementary matrices whose product is A^{-1}.

   (c) Give an example of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbf{R}^3$ such that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is independent, $\{\mathbf{v}_1, \mathbf{v}_3\}$ is independent, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ is independent, but $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is dependent. Explain.

   (d) If A is a matrix, explain why the rows of A are in $N(A)^\perp$. You may not use $R(A) = N(A)^\perp$.

5. Proofs. Do any **two** of the following. If you do more than two, you will get credit for the best two. Use the back of the page if you need more room. (30 points)

   (a) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are non-zero vectors in $\mathbf{R}^n$ that are orthogonal to each other. Prove that they are linearly independent. (Start with the equation we always use to determine independence and dot both sides with $\mathbf{v}_1$.)

   (b) Let $V = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 - x_2 + 2x_3 \leq 0\}$. Prove that V is closed under vector addition and that V is not a subspace of $\mathbf{R}^3$.

   (c) Suppose A is a symmetric matrix, and that $\mathbf{v}$ and $\mathbf{w}$ are vectors such that $A\mathbf{v} = 2\mathbf{v}$ and $A\mathbf{w} = 3\mathbf{w}$. Prove that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

   (d) Let A and B be $k \times \ell$ matrices. Prove that $V = \{\mathbf{x} \in \mathbf{R}^\ell \mid A\mathbf{x} = B\mathbf{x}\}$ is a subspace of $\mathbf{R}^\ell$.

   (e) Suppose V and W are subspaces of $\mathbf{R}^n$ such that $W \subset V$. Prove that $V^\perp \subset W^\perp$.

Partial answers and hints.

1. (a) The rows of R.

   (b) The first, second, and fourth columns of A (the pivot columns).

   (c) $(-2, 1, 1, 0, 0)$ and $(3, 0, 0, 2, 1)$ form a basis.

   (d) 3, 3, 2

3. Five independent vectors need at least five dimensions. An eight dimensional space needs at least eight vectors to span it.

4. (a) You only need to reduce the matrix far enough to see if there will be free variables.

   (b) Remember to reverse the order of the elementary matrices for the inverse.