Math 223 **Final Exam** **Name:**

30 April 2008 200 Points

"Show enough work to justify your answers."

READ CAREFULLY! This exam has **three** parts. You have choices in each part.

Part I. Definitions (50 points). Do **exactly ten** problems in this part. If you do more than ten, I will grade the first ten and ignore the rest. *If you begin one and decide you do not want it graded, cross your work out.*

Define the following terms. Some of the terms may be used in the definitions of other terms. In some cases when a definition involves a formula, the formula by itself is almost never the complete definition. (5 points each)

1. The rank of matrix A is ...
2. A linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is ...
3. The span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is ...
4. $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent if ...
5. A basis for a vector space V is ...
6. The dimension of a vector space V is ...
7. Two vectors $\mathbf{v}$ and $\mathbf{w}$ in an inner product space are orthogonal (perpendicular) if ...
8. A function $f : A \to B$ is one-to-one if ...
9. A function $f : A \to B$ is onto if ...
10. If V and W are vector spaces, a function $T : V \to W$ is linear if ...
11. Define the column space and row space of a matrix A.
12. Define the kernel and nullity of a linear transformation $T : V \to W$.
13. Define the image and rank of a linear transformation $T : V \to W$.
14. Define eigenvector and eigenvalue of a matrix A.
15. The characteristic polynomial of a matrix A is ...

Part II. Computational Problems (75 points). Do **at least five** problems in this part. If you do more than five, you will get credit for the best five. (15 points each)

1. Solve the following linear system. Write the results in parametric form.

$$\begin{array}{ccccccc} 2x & + & y & + & 4z & = & 5 \\ x & & & + & 3z & = & 2 \\ x & + & 3y & - & 3z & = & 5 \end{array}$$

2. Compute the inverse of the following matrix.

$$\begin{pmatrix} 2 & 1 & 1 \\ -1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

3. Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 8 & -6 \\ 9 & -7 \end{pmatrix}$$

4. Given that 12 is an eigenvalue of the matrix A, find a basis for the associated eigenspace $E_A(12)$.

$$A = \begin{pmatrix} 11 & -5 & 3 \\ -2 & 2 & 6 \\ -1 & -5 & 15 \end{pmatrix}$$

5. Let S be the subspace of $\mathbb{R}^3$ given by $2x + y - 2z = 0$. Find an orthogonal basis for S. The basis does not need to be orthonormal.

6. Let S be the subspace of $\mathbb{R}^3$ given by $2x + y - 2z = 0$ (the same as in the previous problem), and let $\mathbf{v} = (-1, 1, 3)$. Find $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_1 \in S$, $\mathbf{v}_2 \perp S$, and $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$.
Note: You may use the answer to the previous problem, but it is possible to solve this without finding a basis for S. Instead, you can project in the direction orthogonal to S.

7. Compute the determinant of the following matrix.

$$\det \begin{pmatrix} 4 & 1 & 2 \\ 0 & -2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

8. Let $\mathbf{v}_1 = (1, 2, -1)$, $\mathbf{v}_2 = (2, -1, 3)$, $\mathbf{w}_1 = (1, 1, 1)$, and $\mathbf{w}_2 = (1, -3, 4)$. Determine if $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2) = \operatorname{span}(\mathbf{w}_1, \mathbf{w}_2)$. One way to do this is to find a basis for $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2)$.

Part III. Theoretical Problems (75 points). Do **at least five** problems in this part. If you do more than five, you will get credit for the best five. (15 points each)

1. Suppose that V is a vector space and that $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. Suppose that one of these vectors is $\mathbf{0}$. Is it possible for $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to be linearly independent? If so, give an example. If not, give a proof.

2. Define $T : \mathbb{C}([-1, 1]) \to \mathbb{R}$ by $T(f) = \int_{-1}^{1} f(x)\, dx$.

   (a) Prove that T is linear. (9 points)

   (b) Find, with verification, two non-zero functions, one of which is in $\ker(T)$ and one of which is not in $\ker(T)$. (6 points)

3. Let A be the matrix below. Let $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (2, 1, 0)$, and $\mathbf{v}_3 = (-3, -3, 2)$.

   (a) Verify that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of A and determine their eigenvalues. (8 points)

   (b) Find the matrix of $\mu_A : \mathbb{R}^3 \to \mathbb{R}^3$ relative to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. (7 points)

   $$A = \begin{pmatrix} 9 & -14 & -6 \\ 9 & -16 & -9 \\ -8 & 16 & 11 \end{pmatrix}$$

4. Let $V = \{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 = y^2 + z^2\}$. Determine, with proof, if V is a subspace of $\mathbb{R}^4$.

5. Define $p_1, p_2, p_3 \in \mathbb{P}_2$ by $p_1(x) = (x - 1)(x - 2)$, $p_2(x) = (x + 1)(x - 2)$, and $p_3(x) = (x + 1)(x - 1)$. Prove that p_1, p_2, and p_3 form a basis of $\mathbb{P}_2$.

6. Let $\{p_1, p_2, p_3\}$ be the basis of $\mathbb{P}_2$ defined in the previous problem. Define $D : \mathbb{P}_2 \to \mathbb{P}_2$ by $D(p) = p'$. Find the matrix of D relative to this basis.
   (Note: If $q(x) = a_1p_1(x) + a_2p_2(x) + a_3p_3(x)$, you can find the coefficients quickly by plugging in -1, 1, and 2 for x.)

7. Suppose that V is a subspace of $\mathbb{R}^{16}$ and that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V. Let $\mathbf{w}_1 = \mathbf{v}_1 + 2\mathbf{v}_2$ and $\mathbf{w}_2 = 3\mathbf{v}_1 - \mathbf{v}_2$.

   (a) Prove that $\mathcal{B}' = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for V. (7 points)

   (b) Find the matrix $C_{\mathcal{B}\mathcal{B}'}$ that converts from coordinates relative to $\mathcal{B}$ to coordinates relative to $\mathcal{B}'$. (8 points)

8. Suppose A is a square matrix. Use mathematical induction to prove that
   $(I - A)\sum_{i=0}^{n} A^i = I - A^{n+1}$ for all $n \geq 1$. Note: $A^0 = I$.