Math 223 **Exam 3** **Name:**

14 April 2008 100 Points

No calculators or *Mathematica*, except as indicated.

"Show enough work to justify your answers."

READ CAREFULLY. This exam has two parts. Read the instructions for each part.

Part I. Do all problems in this part.

1. (15 points) Consider the matrix $A = \begin{pmatrix} 8 & -2 & -28 & -5 & 2 \\ 5 & -2 & -19 & -3 & -1 \\ -13 & 4 & 47 & 10 & 11 \end{pmatrix}$. Its reduced row echelon form is $\begin{pmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}$.

   (a) The linear function μ_A has domain $\mathbf{R}^k$ and range $\mathbf{R}^\ell$, that is, $\mu_A : \mathbf{R}^k \to \mathbf{R}^\ell$. What are the values of k and ℓ?

   (b) Find a basis for $\ker(\mu_A)$.

   (c) Find a basis for the column space of A (this is the same as the image of μ_A) consisting of some of the columns of A.

2. Short Answers. (15 points)

   (a) State what it means for a function $f : X \to Y$ to be one-to-one. (Note: f is not assumed to be linear.)

   (b) If V and W are vector spaces, state what it means for a function $T : V \to W$ to be linear.

   (c) Let $D : \mathbb{P}_3 \to \mathbb{P}_3$ denote differentiation. What is the image of D? (Don't give the definition of the image of a general transformation. Rather, specify what the image is of this particular transformation. One way to do this is to say what functions are in the image of D.)

3. Let $\mathbf{v} = (3, 4)$. Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(\mathbf{x}) = \mathbf{x} - 2\,\mathrm{Proj}_{\mathbf{v}}\,\mathbf{x}$. This is linear (you may assume this). Find the matrix for T relative to the standard basis. (10 points)

Part II. Do **any four** of the problems in this part. If you work on more than four, you will get credit for the best four. (15 points each)

4. Suppose $T : V \to W$ is linear. Suppose that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is an independent subset of W. Prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an independent subset of V.

5. Consider the function $T : \mathbb{P}_2 \to \mathbf{R}^2$ given by $T(p) = \left(p(2) - 2p'(1),\ \int_0^1 p(x)\,dx\right)$. This is linear (you may assume this). Find the matrix for T relative to the bases $\{x^2, x, 1\}$ for $\mathbb{P}_2$ and $\{(1,0),(0,1)\}$ for $\mathbf{R}^2$.

6. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = (1,5)$ and $\mathbf{v}_2 = (0,7)$, and let $\tilde{\mathcal{B}} = \{\mathbf{u}_1, \mathbf{u}_2\}$, where $\mathbf{u}_1 = (2,3)$ and $\mathbf{u}_2 = (-1,2)$. These are ordered bases for $\mathbf{R}^2$.

   (a) Compute the matrix $C_{\mathcal{B}\tilde{\mathcal{B}}}$ that converts coordinates relative to $\mathcal{B}$ into coordinates relative to $\tilde{\mathcal{B}}$. (10 points)

   (b) Let $\mathbf{w} = 8\mathbf{v}_1 - 11\mathbf{v}_2$. Use the matrix $C_{\mathcal{B}\tilde{\mathcal{B}}}$ to compute the coordinates of $\mathbf{w}$ relative to $\tilde{\mathcal{B}}$. (5 points)

7. Suppose $T : V \to V$ is linear. An eigenvector of T is a non-zero vector $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$ for some scalar $\lambda \in \mathbf{R}$. The number λ is called the eigenvalue of T corresponding to $\mathbf{v}$. For example, let V be the vector space of functions that can be differentiated infinitely many times, and consider $D : V \to V$ given by $D(f) = f'$. Then $f(x) = e^{-x}$ is an eigenvector of D with eigenvalue -1 because $D(f) = -f$. Find two more functions g and h that are eigenvectors of D such that f, g, and h are linearly independent, and give their eigenvalues. You do not need to prove that they are independent (but you should be sure that they are).

8. Let $\mathbf{v} \in \mathbf{R}^n$ be a fixed vector. Define $T : \mathbf{R}^n \to \mathbf{R}^n$ by $T(\mathbf{x}) = \mathbf{x} - 2\,\mathrm{Proj}_{\mathbf{v}}\,\mathbf{x}$. Prove that T is linear. Do not assume that $\mathbf{v}$ is the vector in Problem 2.

9. Use the notions of rank and nullity of a linear function to explain the following:

   (a) Why a linear function $T : \mathbf{R}^3 \to \mathbf{R}^5$ cannot be onto, and

   (b) Why a linear function $L : \mathbf{R}^3 \to \mathbf{R}^2$ cannot be one-to-one.