Math 223 **Exam 2** **Name:**

20 March 2008 100 Points

No calculators or *Mathematica*, except as indicated.

"Show enough work to justify your answers."

READ CAREFULLY. This exam has three parts. Read the instructions for each part.

Part I. Do all three problems in this part. (10 points each)

1. Suppose $\mathbf{v}_1 = (2, -1)$ and $\mathbf{v}_2 = (3, 4)$. Find a vector $\mathbf{w}$ in $\mathbf{R}^2$ such that $\mathbf{w} \cdot \mathbf{v}_1 = 2$ and $\mathbf{w} \cdot \mathbf{v}_2 = -3$, find $\mathbf{w}$. Hint: Write $\mathbf{w}$ as $\mathbf{w} = (a, b)$ and use dot products to get information about a and b.

2. Let $\mathbf{v} = (-4, 2)$ and $\mathbf{w} = (3, 1)$. Find vectors $\mathbf{v}_1$, $\mathbf{v}_2$ such that $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1$ is parallel to $\mathbf{v}$, and $\mathbf{v}_2$ is perpendicular to $\mathbf{v}$.

3. The vectors $\mathbf{v}_1 = (1, 1, 1)$ and $\mathbf{v}_2 = (5, 0, 2)$ form an ordered basis for the subspace of $\mathbf{R}^3$ given by $2x + 3y - 5z = 0$ (you do not need to verify this).

   (a) Given that $\mathbf{w} = (7, -3, 1)$ is in that plane (you do not need to verify this), find the coordinates of $\mathbf{w}$ relative to this basis.

   (b) Find the vector in this subspace with coordinates $\binom{1}{3}$.

Part II. Do **any two** of the problems in this part. If you work on more than two, you will get credit for the best two. (15 points each)

4. Given $A = \begin{pmatrix} 1 & 2 & 1 & -2 & -1 \\ 2 & 4 & 3 & 4 & 3 \end{pmatrix}$, we know that the set of solutions of $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbf{R}^5$. Our solution process leads to the solutions being written as

$$(x_1, x_2, x_3, x_4, x_5) = r(-2, 1, 0, 0, 0) + s(10, 0, -8, 1, 0) + t(6, 0, -5, 0, 1).$$

   Prove that the vectors $\mathbf{v}_1 = (-2, 1, 0, 0, 0)$, $\mathbf{v}_2 = (10, 0, -8, 1, 0)$, and $\mathbf{v}_3 = (6, 0, -5, 0, 1)$ form a basis for the subspace.

5. Given $A = (3 \;\; 2 \;\; -1)$ and $B = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 2 & 0 \end{pmatrix}$, show that AB is a linear combination of the rows of B.

6. Let $f(x) = x$ and $g(x) = -x^2$. Find the "angle" between f and g relative to the standard inner product on $C([0, 2])$. Give the exact value of the cosine of the angle, and use a calculator to give an approximation for the angle in degrees or radians (indicate which).

7. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Find three independent eigenvectors of A and their eigenvalues. You should demonstrate that they are eigenvectors, but you do not need to prove that they are independent.

Part III. Do **any two** of the problems in this part. If you work on more than two, you will get credit for the best two. (20 points each)

8. Let $\mathbf{v}_1 = (5, -2, 3)$, $\mathbf{v}_2 = (-1, 3, 2)$, $\mathbf{v}_3 = (7, 5, 12)$.

   (a) Show that $\mathbf{v}_1$, $\mathbf{v}_2$ are independent, that $\mathbf{v}_1$, $\mathbf{v}_2$ are independent, and that $\mathbf{v}_2$, $\mathbf{v}_3$ are independent.

   (b) Are $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ dependent?

   (c) Do $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span $\mathbf{R}^3$? Explain.

9. Let $V = \{p \in \mathbb{P}_4 \mid p'(3) = 0\}$. Let $p_1(x) = 1$, $p_2(x) = (x-3)^2$, $p_3(x) = (x-3)^3$, $p_4(x) = (x-3)^4$.

   (a) Show that p_1, p_2, p_3, p_4 are all in V, and find something in $\mathbb{P}_4$ that is not in V.

   (b) Show that p_1, p_2, p_3, p_4 are linearly independent.

   (c) Explain why $\dim V > 3$ and $\dim V < 5$.

   (d) Do p_1, p_2, p_3, p_4 span V? Explain.

10. Suppose that A is an $n \times n$ matrix, and that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of A with eigenvalue 5.

    (a) Show that $\mathbf{v} + \mathbf{w}$ is an eigenvector of A with eigenvalue 5, assuming $\mathbf{v} + \mathbf{w} \neq \mathbf{0}$.

    (b) If a is a scalar, show that $a\mathbf{v}$ is an eigenvector of A with eigenvalue 5, assuming $a \neq 0$.

    (Note that these prove that the eigenvectors of A with eigenvalue 5, along with the zero vector, form a subspace of $\mathbf{R}^n$.)

11. Let $A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$. Find all eigenvectors of A with eigenvalue 3.