**Math 223** **Exam 1** **Name:**

♡ 14 February 2008 ♡ 100 Points

No *Mathematica*. You may use a calculator to do arithmetic, but exact answers are expected.

*"Show enough work to justify your answers."*

1. Given $A$ below, compute $A^{-1}$ or determine that $A^{-1}$ does not exist. (15 points)

$$A = \begin{pmatrix} -1 & 2 & -2 \\ 1 & -3 & 2 \\ -2 & 6 & -3 \end{pmatrix}$$

2. A matrix and its reduced echelon form are given.

$$\begin{pmatrix} -4 & 11 & -19 & -6 & 24 \\ 1 & -2 & 4 & 2 & -7 \\ -2 & 6 & -10 & -3 & 12 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Given these, what are the solutions of the following systems? Write the solutions in vector form, and as a parameterized line or plane if appropriate. (10 points)

(a)
$$\begin{array}{rcrcrcrcrcl} -4x_1 & + & 11x_2 & - & 19x_3 & - & 6x_4 & + & 24x_5 & = & 0 \\ x_1 & - & 2x_2 & + & 4x_3 & + & 2x_4 & - & 7x_5 & = & 0 \\ -2x_1 & + & 6x_2 & - & 10x_3 & - & 3x_4 & + & 12x_5 & = & 0 \end{array}$$

(b)
$$\begin{array}{rcrcrcrcl} -4x_1 & + & 11x_2 & - & 19x_3 & - & 6x_4 & = & 24 \\ x_1 & - & 2x_2 & + & 4x_3 & + & 2x_4 & = & -7 \\ -2x_1 & + & 6x_2 & - & 10x_3 & - & 3x_4 & = & 12 \end{array}$$

3. Let $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 3 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 & -1 \\ 0 & -2 \end{smallmatrix}\right)$, and $C = \left(\begin{smallmatrix} 5 & 3 \\ 0 & 7 \end{smallmatrix}\right)$. Find scalars $a$ and $b$ such that $aA+bB = C$, or explain why no such scalars exist. (10 points)

4. Consider the following system, in which the constants on the right sides are unspecified. The reduced echelon form of the coefficient matrix (not the augmented matrix) is given.

$$\begin{array}{rcrcrcl} 3x & + & 8y & + & 6z & = & a \\ x & + & 3y & + & 2z & = & b \\ 5x & + & 13y & + & 10z & = & c \end{array} \qquad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Answer the following. (15 points)

(a) Is it possible that the system has no solutions? Briefly explain. (Consider what the reduced augmented matrix might look like.)

(b) Is it possible that the system has exactly one solution? Briefly explain.

(c) Is it possible that the system has infinitely many solutions? Briefly explain.

5. Short answers. (20 points)

(a) Suppose $A$ is a $20\times 30$ matrix and $B$ is a $50\times 20$ matrix. Determine which product is defined, $AB$ or $BA$, and what its dimensions are.

(b) Suppose $S$ is a subset of a vector space $V$. Define what it means for $S$ to be closed under scalar multiplication.

(c) Given pictures of $\mathbf{v}$ and $\mathbf{w}$, draw pictures of $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$. Draw accurately. Use a straightegde if necessary.

$\mathbf{v}+\mathbf{w}$ $\qquad\qquad$ $\mathbf{v}-\mathbf{w}$

(d) Let $A = \left(\begin{smallmatrix} 1 & -1 \\ 3 & -2 \end{smallmatrix}\right)$, and $B = \left(\begin{smallmatrix} 5 & 3 \\ 2 & 1 \end{smallmatrix}\right)$. Compute $AB$ and $BA$. (Note that the two products are not equal.) (5 points)

$AB =$

$BA =$

6. Do any **three** of the following proofs. If you work on more than three, you will get credit for the best three. Continue on the next page if you need more room. (30 points)

(a) Suppose $V$ is a vector space, and that $S$ and $T$ are subspaces of $V$. Prove that $S\cap T$ is a subspace of $V$.

(b) Suppose $A$ is a $6\times 8$ matrix. Then the equation $A\mathbf{x} = \mathbf{0}$ is possible if $\mathbf{x}\in\mathbf{R}^8$ and $\mathbf{0}\in\mathbf{R}^6$ are viewed as column matrices. Let $S = \{\mathbf{x}\in\mathbf{R}^8 \mid A\mathbf{x} = \mathbf{0}\}$. Prove that $S$ is a subspace of $\mathbf{R}^8$.

(c) Suppose $A$ is an $n\times k$ matrix. Prove that the rank of $A$ is less than or equal to both $n$ and $k$.

(d) Let $S = \{(x,y)\in\mathbf{R}^2 \mid x\geq 0 \text{ and } y\geq 0\}$. Determine, with proof, if $S$ is a subspace.

(e) Give a careful proof using the axioms: If $-\mathbf{v} = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$. Justify each step (refer to the axioms by number). The axioms are on the next page.