Math 223 **Final Exam** **Name:**

7 May 2003 200 Points

"Show enough work to justify your answers."

Problems 1–8 are worth 15 points each. You then have a choice any four of the remaining problems, each worth 20 points.

You may use *Mathematica* wherever it is useful, **however**, if its use is essential to your solution, you **must** be very clear how you use it and how you draw conclusions from its results.

1. Definitions.

   (a) Define what it means for $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to be linearly independent.

   (b) Define what the dimension of a vector space is.

   (c) Define what the null space of a matrix A is.

2. Determine if the following vectors are independent or not.

$$\mathbf{v}_1 = (2, -1, 1, 3), \qquad \mathbf{v}_2 = (1, 1, 3, 2), \qquad \mathbf{v}_3 = (1, -5, -7, 1)$$

3. Find all solutions of the following system. Do all work by hand.

$$\begin{array}{rcrcrcrcr} 2w & + & 3x & - & 4y & + & 2z & = & 4 \\ w & & & + & y & + & z & = & 5 \\ -3w & + & 2x & - & 2y & & & = & -2 \end{array}$$

4. Let $\mathbf{v}_1 = (3, 1)$ and $\mathbf{v}_2 = (-2, 2)$. If $\mathbf{w}$ is in $\mathbb{R}^2$, and $\mathbf{w}\cdot\mathbf{v}_1 = 1$ and $\mathbf{w}\cdot\mathbf{v}_2 = 3$, what is $\mathbf{w}$?

5. Find the eigenvalues of $\left(\begin{smallmatrix} 8 & -6 \\ 9 & -7 \end{smallmatrix}\right)$.

6. Given that the eigenvalues of A are 3 and -2, find a basis for each eigenspace of A. $A = \begin{pmatrix} -2 & 10 & 5 \\ 10 & -17 & -10 \\ -20 & 40 & 23 \end{pmatrix}$

7. Let V be the subspace of $\mathbb{R}^3$ defined by $x+11y+4z = 0$. Let $\mathcal{B} = [(3, -5, 13), (-1, 3, -8)]$ and $\tilde{\mathcal{B}} = [(3, -1, 2), (1, 1, -3)]$. These are two bases for V (you don't need to check this). Find the change of basis matrix $C_{\mathcal{B}\tilde{\mathcal{B}}}$.

8. Let V be the subspace of $\mathbb{R}^2$ defined by $y = x$. Define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by letting $L(\mathbf{x})$ be the projection of $\mathbf{x}$ into V. Find the matrix for L relative to the standard basis.

READ THIS !! Do any **four** of the remaining problems. If you work on more than three, you will get credit for the best three. Please do each problem on a separate sheet of paper. Write the problem number in the upper *right* corner of the page and AVOID WRITING ANYTHING IN THE UPPER LEFT CORNER where the staple will go. (20 points each)

9. Find an orthogonal basis for the null space of $\left(\begin{smallmatrix} 2 & 1 & -3 & 2 \\ 3 & 2 & -1 & 1 \end{smallmatrix}\right)$.

10. Consider the vector space $C[-1,2]$ with inner product $f \cdot g = \int_{-1}^{2} f(x)g(x)\,dx$. Find the projection of $f(x) = x^3$ in the direction of $g(x) = x^2$.

11. Suppose that A is a matrix and that AA^T is a diagonal matrix. Explain why the rows of A are orthogonal.

12. Given fixed vectors $\mathbf{v} \in \mathbb{R}^2$ and $\mathbf{w} \in \mathbb{R}^3$, we can define a map $L : \mathbb{R}^2 \to \mathbb{R}^3$ by $L(\mathbf{x}) = (\mathbf{x}{\cdot}\mathbf{v})\mathbf{w}$.
    (a) Prove that L is linear.
    (b) If $\mathbf{v} = (2,3)$ and $\mathbf{w} = (-1,3,2)$, compute the matrix for L.

13. Suppose that $\mathbf{v}$ is an eigenvector for the matrix A with eigenvalue 3. Let $B = A^2 - 2A + 5I$. Show that $\mathbf{v}$ is an eigenvector for B and determine its eigenvalue.

14. Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are non-zero, orthogonal vectors in some vector space. Prove that they are independent.

15. Write $\left(\begin{smallmatrix} 3 & 2 \\ 1 & -1 \end{smallmatrix}\right)$ as a product of elementary matrices.