Math 223 **Final Exam** **Name:** ______________________________

1 May 2001 200 Points

"Show enough work to justify your answers."

You may use *Mathematica* wherever it is useful, **however**, if its use is essential to your solution, you **must** be very clear how you use it and how you draw conclusions from its results.

1. Definitions. (5 points each)
   a) Define what it means for vectors $\mathbf{v}$ and $\mathbf{w}$ to be orthogonal in an inner product space V.
   b) Define what it means for λ to be an eigenvalue for a square matrix A.
   c) Define what the dimension of a vector space V is.
   d) Define what the kernel is of a linear function $L\colon V \to W$.
2. Determine if $x^2 + 3x - 7$, $5x^2 - 4x + 2$, $3x + 2$, and $2x^2 + 7x$ are linearly independent. (15 points)
3. Consider the polynomials $x - 1$ and $x + 3$ in $\mathbb{P}_1$.
   a) Prove that these polynomials form a basis of $\mathbb{P}_1$. (10 points)
   b) Find the coordinates of $2x - 7$ relative to this basis. (10 points)
4. Consider the following matrix A and its reduced form.

$$A = \begin{pmatrix} 2 & 7 & 12 & 30 \\ 4 & 5 & 6 & 24 \\ 1 & 2 & 3 & 9 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   a) Find all solutions of the following system. Express your answer in parametric form. (10 points)

$$\begin{array}{rcrcrcr} 2x & + & 7y & + & 12z & = & 30 \\ 4x & + & 5y & + & 6z & = & 24 \\ x & + & 2y & + & 3z & = & 9 \end{array}$$

   b) Find all solutions of the following system. Express your answer in parametric form. (10 points)

$$\begin{array}{rcrcrcrcr} 2x & + & 7y & + & 12z & + & 30w & = & 0 \\ 4x & + & 5y & + & 6z & + & 24w & = & 0 \\ x & + & 2y & + & 3z & + & 9w & = & 0 \end{array}$$

   c) Find a basis for the row space of A. (5 points)
   d) Find a basis for the column space of A. (5 points)
   e) Find a basis for the kernel of μ_A. (5 points)
5. Determine, with proof, which of the following are vector spaces under the usual definitions of addition and scalar multiplication. (10 points each)
   a) The solutions of the system of equations in part a) of the previous problem.
   b) The set of all functions f in $\mathbb{C}(\mathbb{R})$ such that $\int_0^2 f(x)\,dx = 0$.

6. If A and B are square matrices of the same size, consider the statements $(AB)^t = A^t B^t$ and $(AB)^t = B^t A^t$. One of these statements is an identity (true for all A and B) and the other is not. For the one that is not an identity, give two examples using $2{\times}2$ matrices so that the equation holds for one example but not for the other. (15 points)

7. Let V be the span of $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$. These functions are linearly independent (you don't need to prove this), and so they form a basis for V. Define $D\colon V \to V$ by $D(f) = f'$. Find the matrix of D relative to this basis. (15 points)

READ THIS !! Do any **three** of the remaining problems. If you work on more than three, you will get credit for the best three. Please do each problem on a separate sheet of paper. Write the problem number in the upper right corner of the page and avoid writing anything in the upper left corner. (20 points each)

8. Let V be span$\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$. As in the previous problem, these functions form a basis for V. They are also orthogonal relative to the standard inner product on $\mathbb{C}([-\pi, \pi])$ (you don't need to prove this), *however they are not orthonormal.* Find the projection of $f(x) = x^3 - 9x$ onto V. Write your answer as a linear combination of the basis functions with simplified coefficients.

9. Let U be the subspace of $\mathbb{R}^3$ spanned by $(1, 2, 3)$ and $(2, -1, 0)$. Let V be the subspace of $\mathbb{R}^3$ spanned by $(5, 0, 3)$ and $(0, 5, 6)$. Are U and V the same? Why or why not?

10. Give an example of a function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is not linear and prove that it is not linear.

11. Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is linear. Find the general formula for $T\begin{pmatrix} x \\ y \end{pmatrix}$ given that

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix} \qquad \text{and} \qquad T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}.$$

12. Find the eigenvalues of $\begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$. For each eigenvalue, find a corresponding eigenvector.

Now, go home and don't think about mathematics for a few months! (∞ points)