Math 223 **Exam 1** **Name:**

27 February 2001 100 Points

No *Mathematica*. You may use a calculator to do arithmetic, but exact answers are expected.

"Show enough work to justify your answers."

Read This! Please do each numbered problem on a separate sheet of paper. Put the problem number in the upper **right** corner of the page. Be sure not to write anything near the upper left corner (to leave room for a staple).

1. Definitions and Theorems. (5 points each)
   (a) Give the definition of linear independence of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.
   (b) Give the definition of dimension of a vector space. What fact about vector spaces is important for this definition to make sense?
   (c) State the Comparison Theorem.

2. Examples. (10 points each)
   (a) Give an example of three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in some vector space other than $\mathbf{R}^2$ such that $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, $\mathbf{v}_1, \mathbf{v}_3$ are linearly independent, and $\mathbf{v}_2, \mathbf{v}_3$ are linearly independent, but $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent. You do not need to prove the linear independence/dependence, just give an example.

   (b) Give an example of a function in $\{f \in \mathbb{C}([0,2]) \mid \int_0^2 f(x)\,dx = 0\}$ other than the zero function.

3. Consider the matrix $\begin{pmatrix} 1 & 5 & 1 & 1 & 0 \\ 1 & 5 & 2 & 4 & 1 \\ 2 & 10 & 0 & -4 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{pmatrix}$. Its reduced row echelon form is $\begin{pmatrix} 1 & 5 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
   (a) Given this, what are the solutions of the following systems? Write the solutions in vector form. (7 points each)

   i. $$\begin{array}{rcrcrcrcl} x_1 & + & 5x_2 & + & x_3 & + & x_4 & = & 0 \\ x_1 & + & 5x_2 & + & 2x_3 & + & 4x_4 & = & 1 \\ 2x_1 & + & 10x_2 & + & & - & 4x_4 & = & 0 \\ & & & & x_3 & + & 3x_4 & = & 1 \end{array}$$

   ii. $$\begin{array}{rcrcrcrcrcl} x_1 & + & 5x_2 & + & x_3 & + & x_4 & + & & = & 0 \\ x_1 & + & 5x_2 & + & 2x_3 & + & 4x_4 & + & x_5 & = & 0 \\ 2x_1 & + & 10x_2 & + & & - & 4x_4 & + & & = & 0 \\ & & & & x_3 & + & 3x_4 & + & x_5 & = & 0 \end{array}$$

   (b) The solutions of this second system form a subspace of $\mathbf{R}^5$ (you do not need to prove this). Find a basis for this subspace. Explain how you know it is a basis. (6 points)

4. Let $p_1(x) = x$, $p_2(x) = x(x-1)$, and $p_3(x) = (x+1)^2$. It can be shown that these form a basis for $\mathbb{P}_2$. Let $p(x) = x^2 + x + 1$. Write p as a linear combination of p_1, p_2, and p_3. (10 points)

5. Consider the subspace V of $\mathbf{R}^4$ consisting of solutions of $3x - 2y + 4z - w = 0$.

   (a) Let $\mathbf{v}_1 = (1, 2, 1, 3)$, $\mathbf{v}_2 = (1, 0, -1, -1)$, and $\mathbf{v}_3 = (0, 2, 1, 0)$. Show that these are in V and that they are linearly independent. (13 points)

   (b) Argue that the dimension of V is 3. (5 points)

   (c) Explain why $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for V. (5 points)

6. Do **one** of the following proofs. If you work on both, you will get credit for the best one. (12 points)

   (a) Prove that $V = \{(x, y, z, w) \in \mathbf{R}^4 \mid 3x - 2y + 4z - w = 0\}$ is a vector space.

   (b) Assume that $\mathbf{v}$ is an element of some vector space and that $2\mathbf{v} = \mathbf{v}$. Give a careful proof using the axioms that $\mathbf{v} = \mathbf{0}$. Justify each step of your proof by citing one or more of the axioms.

When you finish, put your pages in order and staple them. Be sure the problem numbers are in the upper right corners of each page, and that the staple does not obscure any of your work.