Math 23 **Final Exam** **Name:**

15 December 1997 200 Points

"Show enough work to justify your answers."

READ CAREFULLY! There are two parts to this exam, each worth 100 points. In the first part you are to work every problem. In the second part you will have a selection.

Part I. There are ten problems in this part, each worth 10 points. Work every problem.

1. Let $\mathbf{v}_1 = (1, 2, -1)$, $\mathbf{v}_2 = (2, -1, 3)$, $\mathbf{w}_1 = (1, 1, 0)$, and $\mathbf{w}_2 = (1, -3, 4)$ Determine if $\text{span}(\mathbf{v}_1, \mathbf{v}_2) = \text{span}(\mathbf{w}_1, \mathbf{w}_2)$.

2. Let $\mathbf{v}_1 = (3, 1)$ and $\mathbf{v}_2 = (-2, 2)$. If $\mathbf{w}\cdot\mathbf{v}_1 = 1$ and $\mathbf{w}\cdot\mathbf{v}_2 = 3$, what is $\mathbf{w}$?

3. Define $D : \mathcal{P}_{100} \to \mathcal{P}_{100}$ by $D(p) = p'$.
   (a) Compute the matrix A of D relative to the standard basis of $\mathcal{P}_{100}$.
   (b) Explain why $A^{100} = 0$.

4. Suppose that A and B are invertible matrices of the same dimension. What is the inverse of AB? Explain.

5. Give an example of a subset of $\mathbb{R}^2$ that is not a subspace, and explain why it is not a subspace.

6. Write $\mathbf{w} = (5, -5, 4)$ as a linear combination of $\mathbf{u} = (2, -1, 2)$ and $\mathbf{v} = (2, 2, -1)$, or show that it is impossible.

7. Give an example of a map $L : \mathbb{R}^2 \to \mathbb{R}^2$ that is not linear, and explain why it is not linear.

8. Given fixed vectors $\mathbf{v} \in \mathbb{R}^2$ and $\mathbf{w} \in \mathbb{R}^3$, we can define a map $L : \mathbb{R}^2 \to \mathbb{R}^3$ by $L(\mathbf{x}) = (\mathbf{x}\cdot\mathbf{v})\mathbf{w}$.
   (a) Prove that L is linear.
   (b) If $\mathbf{v} = (2, 3)$ and $\mathbf{w} = (-1, 3, 2)$, compute the matrix for L.

9. Find a diagonal matrix that is similar to $\begin{pmatrix} 8 & -6 \\ 9 & -7 \end{pmatrix}$.

10. Verify the following handy formula for the inverse of a 2×2 matrix:

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Part II. There are eight problems in this part. Do any **five** of them. If you work on more than five you will get credit for the best five. Please work each problem on a seperate sheet of paper. Put the problem number in the upper right hand corner of the page, and put the pages in order when you turn them in. (20 points each)

11. Suppose that $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are orthogonal, non-zero vectors in $\mathbb{R}^n$, and that $\mathbf{v}$ is a linear combination of them, namely, $\mathbf{v} = a_1\mathbf{w}_1 + \cdots + a_k\mathbf{w}_k$. Derive a formula for the coefficient a_j in terms of $\mathbf{v}$ and $\mathbf{w}_1, \ldots, \mathbf{w}_k$.

12. Let V and W be vector spaces and let $L : V \to W$ be linear. Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ and that $L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)$ are linearly independent. Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

13. Consider the non-homogeneous equation $A\mathbf{x} = \mathbf{b}$ and its associated homogeneous equation $A\mathbf{x} = \mathbf{0}$. The point of this problem is to see how the solutions of these are related.

    (a) Show that if $\mathbf{p}$ and $\mathbf{q}$ are solutions of the non-homogeneous equation, then $\mathbf{x} = \mathbf{p} - \mathbf{q}$ is a solution of the homogeneous equation.

    (b) Show that if $\mathbf{p}$ is a solution of the non-homogeneous equation and $\mathbf{h}$ is a solution of the homogeneous equation, then $\mathbf{x} = \mathbf{p} + \mathbf{h}$ is a solution of the non-homogeneous equation.

    (c) Parts a) and b) imply the following: If $\mathbf{p}$ is a *particular* solution of the non-homogeneous equation and $\mathbf{h}$ represents the *general* solution of the homogeneous equation (that is, anything in the null space of A), then the *general* solution of the non-homogeneous equation is $\mathbf{x} = \mathbf{p} + \mathbf{h}$. Illustrate this with the example where the reduced form of $(A|\mathbf{b})$ is the following matrix, that is, write the solution as $\mathbf{x} = \mathbf{p} + \mathbf{h}$ where $\mathbf{p}$ is a particular (constant) vector and $\mathbf{h}$ is a linear combination of basis vectors for the null space of A. $\begin{pmatrix} 1 & 0 & 2/5 & 1/5 & 1/5 \\ 0 & 1 & 6/5 & -7/5 & -2/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

14. Consider $V = \{X \mid X \text{ is a } 2{\times}2 \text{ matrix and } \operatorname{tr} X = 0\}$.

    (a) Prove that V is a vector space.

    (b) Find a basis for V.

15. Suppose that $\mathbf{v}$ is an eigenvector for the matrix A with eigenvalue 3. Let $B = A^2 - 2A + 5I$. Show that $\mathbf{v}$ is an eigenvector for B and determine its eigenvalue.

16. A matrix A is called *nilpotent* if $A^k = 0$ for some sufficiently large k, that is, if you multiply A by itself enough times you get the zero matrix. Prove that if A is nilpotent, then the only eigenvalue it can have is 0. Hint: What happens if λ is a non-zero eigenvalue?

17. Suppose that M is a given square matrix. The point of this problem is to show that M can be written as a sum of a symmetric matrix and an antisymmetric matrix in a unique way.

    (a) Let $S = \frac{1}{2}(M + M^T)$ and $A = \frac{1}{2}(M - M^T)$. Show that S is symmetric, that A is antisymmetric matrix ($A^T = -A$), and that $M = S + A$.

    (b) Suppose that $M = \tilde{S} + \tilde{A}$ where $\tilde{S}$ is symmetric and $\tilde{A}$ is antisymmetric, perhaps different from S and A. Prove that $\tilde{S}$ and $\tilde{A}$ actually have to be S and A. Hint: Caution! Don't start with the thing you want to conclude. To show that $\tilde{S}$ has to equal S, add $M = \tilde{S} + \tilde{A}$ to its transpose.

    (c) Illustrate this decomposition by writing $\left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$ as the sum of symmetric and antisymmetric matrices.

18. I claim that the following is an inner product on $\mathcal{P}_4$:

$$\langle p, q \rangle = p(0)q(0) + p'(0)q'(0) + p''(0)q''(0) + p'''(0)q'''(0).$$

    (a) In order to be sure this is an inner product, one would need to prove that it satisfies all of the properties of an inner product. Prove the property that if $\langle p, p \rangle = 0$, then $p = 0$, where 0 represents the zero polynomial.

    (b) Find an orthonormal basis for $\mathcal{P}_4$ relative to this inner product.

    (c) Find the projection of $f(x) = e^x$ onto $\mathcal{P}_4$ relative to this inner product. Do you recognize the answer?