Math 23 **Exam 3** **Name:**

25 November 1997 100 Points

No *Mathematica*. You may use a calculator to do arithmetic, but exact answers are expected.

"Show enough work to justify your answers."

READ CAREFULLY! There are nine problems. Do any **six** of them. If you work on more than six you will get credit for the best six. I suggest reading through the entire exam before you start on any of the problems. The problems are worth 15 points each, and 10 points are free!

1. Let $p(x) = 2x - 3$ and $q(x) = x^2$. Compute $\text{Proj}_p q$, the projection of q in the direction of p, relative to the inner product $\langle f, g\rangle = \int_{-2}^{2} f(x)g(x)\,dx$.

2. Suppose that $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are orthonormal vectors in $\mathbb{R}^n$, and that $\mathbf{v}$ is a linear combination of them, namely, $\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k$. Prove that the coefficient a_j is $\mathbf{v} \cdot \mathbf{u}_j$.

3. The vectors

$$\mathbf{u}_1 = (1/5, 2/5, 2/5, 4/5), \qquad \mathbf{u}_2 = (-2/5, 1/5, -4/5, 2/5),$$
$$\mathbf{u}_3 = (-4/5, 2/5, 2/5, -1/5), \quad \mathbf{u}_4 = (2/5, 4/5, -1/5, -2/5)$$

form an orthonormal basis of $\mathbb{R}^4$.

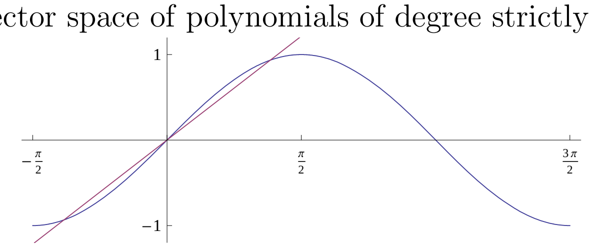
   (a) Verify that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthonormal. (5 points)

   (b) Let $V = \text{span}(\mathbf{u}_1, \mathbf{u}_2)$ and $W = \text{span}(\mathbf{u}_3, \mathbf{u}_4)$. Find the matrix P (with respect to the standard basis) of orthogonal projection from $\mathbb{R}^4$ to V. (10 points)

   (c) **Extra Credit.** Without computing it, explain why the matrix Q of orthogonal projection from $\mathbb{R}^4$ to W should be $I - P$, where I is the identity matrix. (5 points)

4. The picture shows the graphs of the sine function and the closest polynomial in $\mathcal{P}_2$ relative to the inner product $\langle f, g\rangle = \int_{-\pi/2}^{\pi/2} f(x)g(x)\,dx$. Add to the picture the graph (approximately) of the polynomial in $\mathcal{P}_2$ closest to the sine function relative to the inner product $\langle f, g\rangle = \int_{\pi/2}^{3\pi/2} f(x)g(x)\,dx$. (Don't compute this, just eyeball it!) Explain in your own words why these two "closest lines" are different. In what sense are they closest?

   Note: $\mathcal{P}_2$ is the vector space of polynomials of degree strictly less than 2.



5. An *orthogonal matrix* is a square matrix A such that $A^{-1} = A^T$. Prove that an $n \times n$ matrix is orthogonal if and only if its columns form an orthonormal basis of $\mathbb{R}^n$. Hint: Think about the product $A^T A$ and about how matrix multiplication and dot products are related.

6. Suppose A is a matrix. Explain why RowSpace(A) and null(A) are orthogonal.

   Note: null(A) is another notation for ker(A).

7. Let A be the following matrix. Its row reduced form E is also shown.

$$A = \begin{pmatrix} 3 & -5 & 16 & -4 \\ 2 & 7 & -10 & 18 \\ 1 & 2 & -2 & 6 \\ 2 & -1 & 6 & 2 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) Find an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^4$ so that the first vectors of $\mathcal{B}$ are a basis for RowSpace(A), and and the last vectors of $\mathcal{B}$ are a basis for null(A). (Note that these subspaces are orthogonal by the previous problem.)

   (b) Let $\mathbf{v} = (1, 2, 2, 1)$. Decompose $\mathbf{v}$ into the sum of two vectors, one of which is in RowSpace(A) and the other of which is in null(A). That is, find vectors $\mathbf{w}_1 \in$ RowSpace(A) and $\mathbf{w}_2 \in$ null(A) so that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$.

8. Let $\mathbf{v} = (1, 1)$. (5 points each)

   (a) Find a non-zero vector $\mathbf{w}$ in $\mathbb{R}^2$ that is orthogonal to $\mathbf{v}$ relative to the standard inner product. Draw both vectors on the same graph.

   (b) For $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$, the formula $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 4 x_2 y_2$ is an inner product on $\mathbb{R}^2$ (you don't need to verify this). What is the angle (approximately, in degrees) between $\mathbf{v}$ and $\mathbf{w}$ relative to this inner product?

   (c) Find a non-zero vector $\mathbf{u}$ in $\mathbb{R}^2$ that is orthogonal to $\mathbf{v}$ relative to this non-standard inner product. Add $\mathbf{u}$ to the graph with the other two vectors.

9. Let $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2]$, where $\mathbf{v}_1 = (1, 5)$ and $\mathbf{v}_2 = (0, 7)$, and let $\tilde{\mathcal{B}} = [\mathbf{u}_1, \mathbf{u}_2]$, where $\mathbf{u}_1 = (2, 3)$ and $\mathbf{u}_2 = (-1, 2)$.

   Note: $\mathcal{B} = [\mathbf{v}_1, \mathbf{v}_2]$ is notation for an ordered basis.

   (a) Compute the matrix $C_{\mathcal{B}\tilde{\mathcal{B}}}$ that converts coordinates relative to $\mathcal{B}$ into coordinates relative to $\tilde{\mathcal{B}}$. (10 points)

   (b) Let $\mathbf{w} = 8\mathbf{v}_1 - 11\mathbf{v}_2$. Use the matrix $C_{\mathcal{B}\tilde{\mathcal{B}}}$ to compute the coordinates of $\mathbf{w}$ relative to $\tilde{\mathcal{B}}$. (5 points)