Math 23 **Exam 2** **Name:**

28 October 1997 100 Points

No *Mathematica*. You may use a calculator to do arithmetic, but exact answers are expected.

"Show enough work to justify your answers."

All problems are worth 10 points.

1. Suppose $A\mathbf{x} = \mathbf{b}$ is a consistent system of equations. What is the relationship between the number of free variables in this system and $\dim N(A)$? Explain.

2. A matrix A and its echelon reduced form U are given. Based on this, find a basis for the null space of A.

$$A = \begin{pmatrix} 3 & -1 & 0 & 2 & 1 \\ 2 & 1 & 2 & -1 & 0 \\ 5 & 0 & 2 & 1 & 1 \\ 1 & -2 & -2 & 3 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 & 2/5 & 1/5 & 1/5 \\ 0 & 1 & 6/5 & -7/5 & -2/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3. Let V be the vector space of functions that satisfy $f'' + f = 0$. We have seen that $\mathcal{B} = [S, C]$ form a basis for V, where $S(x) = \sin x$ and $C(x) = \cos x$. Define $D : V \to V$ by $D(f) = f'$. Write the matrix for D relative to the basis $\mathcal{B}$. (Note: The same basis is used for both domain and range.)

4. Let V be the span of $(2, -1, 0, 3, -2)$, $(-1, 3, 2, 0, 1)$, $(5, 0, 2, 9, -5)$, and $(3, -2, 4, 1, 1)$. Determine some collection of these vectors that form a basis of V. What is the dimension of V?

5. Write $\mathbf{w} = (5, -5, 4)$ as a linear combination of $\mathbf{u} = (3, -1, 2)$ and $\mathbf{v} = (2, 2, 3)$, or show that it is impossible.

6. A linear map $L : \mathbb{R}^2 \to \mathbb{R}$ has the properties that $L(2, 3) = 5$ and $L(-1, 2) = 3$. Find the formula for $L(x, y)$.

7. Give examples (by formulas or pictures) of the following and briefly explain.

   (a) A subset of $\mathbb{R}^2$ that is closed under addition but not under scalar multiplication.

   (b) A subset of $\mathbb{R}^2$ that is closed under scalar multiplication but not under addition.

READ CAREFULLY!! Do any **THREE** of the following. If you work on more than three, you will get credit for the best three.

8. If the following statement is true, prove it. If it is false, give a counterexample. "If $\mathbf{u}$ and $\mathbf{v}$ are independent, and $\mathbf{v}$ and $\mathbf{w}$ are independent, and $\mathbf{w}$ and $\mathbf{u}$ are independent, then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are independent."

9. Let V be the collection of all 2×2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $a + d = 0$. Determine, with proof, if V is a subspace of $\mathbb{R}^{2\times 2}$.

10. Suppose that A is a matrix and that $\dim N(A) \geq 1$. Prove that the columns of A are linearly dependent.

11. Suppose $L : V \to W$ is linear. Prove that $\ker L$ is a subspace of V.

12. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (2x + y, xy)$. Determine, with proof, if T is linear.