Math 23 **Final Exam** **Name:**

16 December 1994 200 Points

"Show enough work to justify your answers."

1. Definitions. (5 points each)
   (a) Define what it means for $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to be linearly dependent.

   (b) Define what it means for $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to span a vector space V.

   (c) Define what it means for a function $f : A \to B$ to be one-to-one, where A and B are sets.

   (d) Define what it means for a function $f : A \to B$ to be onto.

   (e) Define what it means for a subset S of a vector space to be closed under scalar multiplication.

   (f) Define what the kernel is of a linear function $L : V \to W$.

2. Determine if $x^2 + 3x - 7$, $5x^2 - 4x + 2$, $3x + 2$, and $2x^2 + 7x$ are linearly independent. (10 points)

3. Consider the polynomials $x - 1$ and $x + 3$ in $\mathbb{P}_1$.
   (a) Prove that these polynomials form a basis of $\mathbb{P}_1$. (10 points)

   (b) Find the coordinates of $2x - 7$ relative to this basis. (10 points)

4. Suppose that A is an $n \times n$ matrix and that $A^T A = I$. Prove that the columns of A form an orthonormal basis for $\mathbb{R}^n$. (15 points)

5. Consider the following linear system and matrix.

$$\begin{array}{ccccccc} 2x & + & 7y & + & 12z & = & 30 \\ 4x & + & 5y & + & 6z & = & 24 \\ x & + & 2y & + & 3z & = & 9 \end{array} \qquad \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) Given that the matrix is the reduced row echelon form of the augmented coefficient matrix of the system, find all solutions of the system. (10 points)

   (b) What are the solutions of the corresponding homogeneous system? (10 points)

6. Determine, with proof, which of the following are vector spaces under the usual definitions of addition and scalar multiplication. (10 points each)

   (a) The solutions of the system of equations in the previous problem.

   (b) The set of all functions f in $\mathbb{C}(\mathbb{R})$ such that $\int_0^2 f(x)\,dx = 0$.

7. Let U be the subspace of $\mathbb{R}^3$ spanned by $(1, 2, 3)$ and $(2, -1, 0)$. Let V be the subspace of $\mathbb{R}^3$ spanned by $(5, 0, 3)$ and $(0, 5, 6)$. Are U and V the same? Why or why not? (15 points)

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be orthogonal projection onto the line $y = x$. Find the matrix of T relative to the basis $\{(1, 1), (1, -1)\}$. (10 points)

9. Give an example of a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ that is not linear and explain why it is not linear. (15 points)

10. Suppose that **u**, **v**, and **w** are non-zero orthogonal vectors in some inner product space. Prove that they are linearly independent. (15 points)

11. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is linear. Find the general formula for $T\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ given that

$$T\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 5 \\ -3 \\ -1 \end{smallmatrix}\right) \qquad \text{and} \qquad T\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 \\ -3 \\ 2 \end{smallmatrix}\right).$$

    (15 points)

12. Find the eigenvalues and eigenvectors of $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. (15 points)

13. Go home and don't think about mathematics for a few weeks! (∞ points)