Math 23 **Exam 2** **Name:**

21 November 1994 100 Points

No *Mathematica*. You may use a calculator to do arithmetic, but exact answers are expected.

"Show enough work to justify your answers."

1. Short Answers. (5 points each)
   (a) State the Cauchy-Schwarz inequality.
   (b) State what it means for a function $f : X \to Y$ to be one-to-one.
   (c) Given the pictures of $\mathbf{v}$ and $\mathbf{w}$, add $\mathrm{Proj}_{\mathbf{v}}\mathbf{w}$ to the picture.
   (d) Describe the kernel of $D : \mathbb{P}_3 \to \mathbb{P}_3$, where D denotes differentiation.
   (e) Describe the image of $D : \mathbb{P}_3 \to \mathbb{P}_3$, where D denotes differentiation.

2. Find an orthogonal basis for the subspace of $\mathbb{R}^3$ of solutions of $3x - 2y + z = 0$. (10 points)

3. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (3x - y, 2x)$. This is linear and invertable (you may assume this). Find a formula for T^{-1}. (10 points)

4. Consider the matrix $A = \begin{pmatrix} 1 & 5 & 1 & 1 & 0 \\ 1 & 5 & 2 & 4 & 1 \\ 2 & 10 & 0 & -4 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{pmatrix}$. Its reduced row echelon form is $\begin{pmatrix} 1 & 5 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
   (a) The linear function μ_A has domain $\mathbb{R}^k$ and range $\mathbb{R}^\ell$, that is, $\mu_A : \mathbb{R}^k \to \mathbb{R}^\ell$. What are the values of k and ℓ? (5 points)
   (b) Find a basis for the vector space of solutions of $A\mathbf{x} = \mathbf{0}$. (10 points)
   (c) Find a basis for the row space of A. (5 points)

5. Consider the function $T : \mathbb{P}_2 \to \mathbb{R}^2$ given by $T(p) = \left(\int_0^1 p(x)\,dx,\ p'(2)\right)$. This is linear (you may assume this). Find the matrix for T relative to the bases $\{x^2, x, 1\}$ for $\mathbb{P}_2$ and $\{(1, 0), (0, 1)\}$ for $\mathbb{R}^2$. (10 points)

6. For $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ in $\mathbb{R}^2$, let $\langle \mathbf{v}, \mathbf{w} \rangle = 2v_1w_1 - v_1w_2 - v_2w_1 + 5v_2w_2$. This is an inner product (you may assume this). Find the line in $\mathbb{R}^2$ through the origin that is orthogonal to the y-axis relative to this inner product (the y-axis is spanned by $(0, 1)$, of course). A sufficient answer is either a vector that spans the line or an equation for the line. (10 points)

7. Suppose that A is a fixed $n \times n$ matrix, and that $L : \mathbb{M}(n, n) \to \mathbb{M}(n, n)$ is defined by $L(X) = AX - XA$.

   (a) Prove that L is linear. (10 points)

   (b) Suppose that $n = 2$ and $A = \left(\begin{smallmatrix} 1 & 3 \\ -2 & 0 \end{smallmatrix}\right)$.

      i. Compute $L\left(\left(\begin{smallmatrix} 3 & -1 \\ 1 & 2 \end{smallmatrix}\right)\right)$. (5 points)

      ii. **Extra Credit.** (Do on back.) Find the matrix for L relative to the basis

      $$\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right\}. \qquad \text{(5 points)}$$

      iii. **Extra Credit.** Find two independent elements of $\ker L$. (5 points)

8. Have a Happy Turkey Day!