Math 23 **Exam 1** **Name:**

7 October 1994 100 Points

"Show enough work to justify your answers."

1. Definitions and Theorems. (5 points each)

   (a) Give the definition of linear independence of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

   (b) Give the definition of dimension of a vector space. What fact about vector spaces is important for this definition to make sense?

   (c) Let S be a subset of a vector space V. Define what it means for S to be closed under vector addition.

   (d) State the Comparision Theorem.

2. Given the pictures of $\mathbf{v}$ and $\mathbf{w}$, draw the pictures of $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ as indicated. (10 points)

   $\mathbf{v}+\mathbf{w}$ $\mathbf{v}-\mathbf{w}$

3. Consider the matrix $\begin{pmatrix} 1 & 5 & 1 & 1 & 0 \\ 1 & 5 & 2 & 4 & 1 \\ 2 & 10 & 0 & -4 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{pmatrix}$. Its reduced row echelon form is $\begin{pmatrix} 1 & 5 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

   (a) Given this, what are the solutions of the following systems? Write the solutions in vector form. (Take advantage of the similarities of the systems.) (5 points each)

   i.
$$\begin{array}{rcrcrcrcl} x_1 & + & 5x_2 & + & x_3 & + & x_4 & = & 0 \\ x_1 & + & 5x_2 & + & 2x_3 & + & 4x_4 & = & 1 \\ 2x_1 & + & 10x_2 & + & & - & 4x_4 & = & 0 \\ & & & & x_3 & + & 3x_4 & = & 1 \end{array}$$

   ii.
$$\begin{array}{rcrcrcrcrcl} x_1 & + & 5x_2 & + & x_3 & + & x_4 & + & & = & 0 \\ x_1 & + & 5x_2 & + & 2x_3 & + & 4x_4 & + & x_5 & = & 0 \\ 2x_1 & + & 10x_2 & + & & - & 4x_4 & + & & = & 0 \\ & & & & x_3 & + & 3x_4 & + & x_5 & = & 0 \end{array}$$

   (b) The solutions of this second system form a subspace of $\mathbf{R}^5$ (you do not need to prove this). Find a basis for this subspace. Explain how you know it is a basis. (5 points)

4. Suppose that S and T are subspaces of a vector space V. Prove that $S \cap T$ is also a subspace of V. (15 points)

5. Consider the polynomials $x - 1$ and $x + 3$ in $\mathbb{P}_1$.

   (a) Prove that these polynomials form a basis of $\mathbb{P}_1$. (8 points)

   (b) Find the coordinates of $2x - 7$ relative to this basis. (7 points)

6. Give an example of three non-zero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in some vector space other than $\mathbf{R}^2$ such that $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, $\mathbf{v}_1, \mathbf{v}_3$ are linearly independent, and $\mathbf{v}_2, \mathbf{v}_3$ are linearly independent, but $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent. You do not need to prove the linear independence/dependence, just give an example. (10 points)

7. Give a careful proof using the axioms: If $2\mathbf{v} = \mathbf{v}$ then $\mathbf{v} = \mathbf{0}$. Justify each step of your proof by citing one or more of the axioms. (15 points) (Note: The list of axioms was included on the exam.)