A CONCRETE PROBLEM WORKED ABSTRACTLY: ANGLES IN A TETRAHEDRON USING A COORDINATE-FREE APPROACH

Math 223

Problem. Let C be the center of a regular tetrahedron in $\mathbb{R}^3$. Find the angle between two of the segments joining C to the vertices.

This is problem 4 in Section 4.2 of our text (Messer). I almost put it on a problem set, but then I realized that it's pretty hard. The reason it's hard is because the easy way to do it would probably not occur to you.

You could waste a lot of time trying to figure out coordinates for the vertices and the center. Once you have those, of course, then the problem is easy.

Let the vertices of the tetrahedron be A, B, D, and E. Instead of determining specific coordinates (which would be position vectors) for the vertices, a better approach is to specify *properties* that the vectors must satisfy, and then work the problem based on those properties. This is more abstract, but it gives a much cleaner solution to the problem.

We can assume that A is the origin. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the position vectors of B, D, and E. As line segments, these are the three edges of the tetrahedron that meet at A. The other three edges are given by the pairwise differences of these vectors.

What has to be true? A regular tetrahedron is made up of four equilateral triangles. Thus $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ all have to have the same length, namely, the length of the sides of the tetrahedron. Let's call that ℓ. Thus

$$(1) \qquad \|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \|\mathbf{v}_3\| = \ell, \quad \text{which is the same as} \quad \mathbf{v}_1 \cdot \mathbf{v}_1 = \mathbf{v}_2 \cdot \mathbf{v}_2 = \mathbf{v}_3 \cdot \mathbf{v}_3 = \ell^2.$$

In addition, the angles between each pair of them must be 60°. That can be specified by using (drum roll please) dot products. The most important formula concerning dot products (and more generally inner products) is $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta$, where θ is the angle between the vectors. Thus we have

$$(2) \qquad \mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = \ell^2 \cos 60^\circ = \frac{\ell^2}{2}.$$

(The second-most important formula concerning dot products is $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.)

I claim that the position vector of the center C of the tetrahedron is $\mathbf{w} = \frac{1}{4}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$. This is, in fact, the average of the vertices—remember one vertex is the origin. (Insightful observation: This is really using $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as a basis for $\mathbb{R}^3$ that is adapted to the problem, and writing $\mathbf{w}$ in terms of this basis.) Then the vector from C to A is $-\mathbf{w}$, and the vector from C to B is $\mathbf{v}_1 - \mathbf{w}$. If α is the desired angle, then

$$(-\mathbf{w}) \cdot (\mathbf{v}_1 - \mathbf{w}) = \|-\mathbf{w}\| \, \|\mathbf{v}_1 - \mathbf{w}\| \cos\alpha = \|\mathbf{w}\| \, \|\mathbf{v}_1 - \mathbf{w}\| \cos\alpha. \tag{3}$$

Expanding the left side of this and using (1) and (2) we have

$$\begin{aligned}
(-\mathbf{w}) \cdot (\mathbf{v}_1 - \mathbf{w}) &= \mathbf{w} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v}_1 \\
&= \frac{1}{16}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \cdot (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) - \frac{1}{4}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \cdot \mathbf{v}_1 \\
&= \frac{1}{16}(\mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 + \mathbf{v}_3 \cdot \mathbf{v}_3 + 2\mathbf{v}_1 \cdot \mathbf{v}_2 + 2\mathbf{v}_1 \cdot \mathbf{v}_3 + 2\mathbf{v}_2 \cdot \mathbf{v}_3) \\
&\qquad - \frac{1}{4}(\mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_1 + \mathbf{v}_3 \cdot \mathbf{v}_1) \\
&= \frac{1}{16}(6\ell^2) - \frac{1}{4}(2\ell^2) = \frac{3}{8}\ell^2 - \frac{1}{2}\ell^2 = -\frac{1}{8}\ell^2.
\end{aligned}$$

If you chase through part of this computation, you see that $\mathbf{w} \cdot \mathbf{w} = 3\ell^2/8$, and so $\|\mathbf{w}\| = \ell\sqrt{3/8}$. A similar computation yields $\|\mathbf{v}_1 - \mathbf{w}\| = \ell\sqrt{3/8}$. Then (3) becomes $-\ell^2/8 = (3\ell^2/8)\cos\alpha$. Thus

$$\cos\alpha = -\frac{1}{3}, \qquad \text{or} \qquad \alpha = \arccos\left(-\frac{1}{3}\right) \approx 1.91 \,\text{radians} = 109.5^\circ.$$

One final observation ... there was no need to assume that one of the vertices is the origin. Instead of $\mathbf{v}_1$ being the position vector of B, it is the difference $B - A$, so it's still the vector from A to B, which I like to write as $\overrightarrow{AB}$. Similarly, $\mathbf{v}_2 = \overrightarrow{AD} = D - A$, $\mathbf{v}_3 = \overrightarrow{AE} = E - A$, and $\mathbf{w} = \overrightarrow{AC} = C - A$. The rest of the computation is unchanged. Note that if you solve the last equation for C and substitute, you get

$$C = A + \mathbf{w} = A + \frac{1}{4}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \frac{1}{4}(A + B + D + E),$$

which more nicely represents the center as the average of the vertices.

Note that when we think of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{w}$ in this way, that is, as displacement vectors from A and not as position vectors, then A, B, C, D, and E represent position vectors from some *completely unspecified* origin. Think about this. The vector A could be any vector whatsoever, depending on where the origin is. The same is true about B, C, D, and E. However the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{w}$ are not arbitrary at all. They would be the same no matter where the origin is.

My feeling about this is that if viewing A, B, C, D, and E as vectors (position vectors) requires some arbitrary sixth point to be an origin, then maybe it's better to think of them as points instead of vectors. (After all, look around you. Do you see an origin? No.) Vectors still exist, but they are displacements between points. Thus, even though A and B are not vectors, $\mathbf{v}_1$ is still $\overrightarrow{AB}$, the vector from A to B. There is a way to make this point of view rigorous, you can still write $\overrightarrow{AB}$ as $B - A$ and the algebra all works out. (Three-dimensional space becomes an *affine space* instead of a vector space.) There's also a benefit for doing this. In most geometric problems that involve vectors (many physics problems, for example), the origin is either some point related to the problem or it is completely arbitrary. In either case, it is just a convenient reference point, and it often gets in the way. If you don't worry about where the origin is and treat the points as points instead of as position vectors, the problem often becomes simpler.

One last thing, and then I really will quit ... as dependent as *Mathematica* is on coordinates, it's possible to write *Mathematica* code in a largely coordinate-free way. I really like this because it allows me to write *Mathematica* code in the way I think mathematically.

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