

## USES OF EIGENVALUES AND EIGENVECTORS

Math 223

In the chapter on eigenvectors and eigenvalues, nearly every linear algebra text has a section on symmetric matrices. This because symmetric matrices occur in many applications of linear algebra and they have particularly nice eigenvectors and eigenvalues.

**Theorem.** *Let  $A$  be an  $n \times n$  symmetric matrix. Then*

- *The eigenvalues of  $A$  are all real.*
- *$\mathbb{R}^n$  has an orthogonal basis consisting of eigenvectors of  $A$ .*

You already know that orthogonal bases are easier to use than arbitrary bases, and so this theorem says that  $\mathbb{R}^n$  has a nice basis consisting of vectors that are particularly important for the matrix  $A$ .

### CRITICAL POINTS FOR FUNCTIONS OF SEVERAL VARIABLES

Let  $f$  be a function of two variables,  $z = f(x, y)$ . Suppose  $(a, b)$  is a critical point, that is, a point where the partial derivatives  $f_x(a, b)$  and  $f_y(a, b)$  are both zero. In Math 112 you learned the Second Derivative Test, which determines if the critical point is a local minimum, a local maximum, or a saddle point. This test involves the quantity  $D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$ , which is the determinant of the matrix

$$f''(a, b) = \begin{pmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{pmatrix}.$$

(Recall that the mixed partial derivatives  $f_{xy}$  and  $f_{yx}$  are equal for reasonable functions. Because of this,  $f''(a, b)$  is a symmetric matrix.) Note that the second derivative  $f_{xx}(a, b)$  measures the concavity of the graph in the  $x$ -direction and  $f_{yy}(a, b)$  measures the concavity of the graph in the  $y$ -direction. There is a way to measure the concavity of the graph in any direction (some kind of second derivative in that direction).

One can ask in which directions the concavity is greatest and least, and what the measure is of those concavities (second derivatives in those particular directions). The answer to this question is given by the eigenvectors and eigenvalues of the matrix  $f''(a, b)$ . More specifically, the directions of maximum and minimum concavity are given by the eigenvectors of this matrix. If  $\mathbf{v}$  is one of the eigenvectors and the corresponding eigenvalue is  $\lambda$ , then  $\lambda$  is the second derivative of  $f$  at  $(a, b)$  in the direction of  $\mathbf{v}$ , and so  $\lambda$  is the measure of the concavity in that direction. The eigenvectors for distinct eigenvalues of a symmetric matrix are orthogonal, and so the directions of maximum and minimum concavity are perpendicular.

This fact has applications in concrete max/min problems—if you are at a local minimum there may be significance to knowing which direction to move in order to increase

most quickly or most gradually. If you are at a saddle point, it's more pronounced—the eigenvectors tell you in which directions the function increases and decreases most quickly.

This significance of the eigenvalues also leads to a proof of the Second Derivative Test. The determinant of a matrix is the product of its eigenvalues. Thus, if  $D < 0$ , then one eigenvalue is positive and one is negative. Consequently the critical point is a saddle point. On the other hand, if  $D > 0$ , then either both eigenvalues are positive or both are negative. If both are positive, then the function is concave up in all directions. If both are negative, then the function is concave down in all directions. Checking the sign of  $f_{xx}$  determines the concavity in the  $x$ -direction, and hence all directions.

These properties of eigenvectors and eigenvalues are true for functions of more variables as well.

### MOMENT OF INERTIA TENSOR OF A SOLID

Suppose you have a solid object, say a book or a tennis racquet, and you toss it in the air. Just as you let go, give it a twist so that it spins. If you experiment, you will find that it spins smoothly around some axes, but around others it wobbles. (Put a rubber band around the book to keep it from opening while spinning.)

Given a solid object, there is a  $3 \times 3$  symmetric matrix  $M$  associated with it, called its moment of inertia tensor. The matrix is determined by the distribution of mass in the object and a coordinate system attached to the object. The eigenvectors (which are orthogonal) represent the axes about which the object will spin without wobbling and are fixed to the object. If  $\boldsymbol{\omega}$  is the angular velocity and  $\mathbf{L}$  is the angular momentum of the object, then  $\mathbf{L} = M\boldsymbol{\omega}$ . The vector  $\boldsymbol{\omega}$  is the axis of rotation and its magnitude is the speed of rotation.

For a freely-spinning object, say your tennis racquet or a satellite, the angular momentum vector  $\mathbf{L}$  is constant in space. Now think about this—the object is rotating around the axis  $\boldsymbol{\omega}$ , the vector  $\mathbf{L}$  is constant, and  $\boldsymbol{\omega}$  and  $\mathbf{L}$  are related by  $\mathbf{L} = M\boldsymbol{\omega}$ . If  $\boldsymbol{\omega}$  and  $\mathbf{L}$  aren't parallel, the object has to wobble! If  $\boldsymbol{\omega}$  and  $\mathbf{L}$  are parallel (which says that  $\boldsymbol{\omega}$  is an eigenvector of  $M$ ), then the object can spin without wobbling. For an object that is constrained to spin about a particular axis, such as a turbine or fan, if  $\boldsymbol{\omega}$  is not an eigenvector of  $M$ , then the angular momentum has to rotate, which has to be caused by a force perpendicular to the axis of rotation. The result is that the mount holding the object shakes or that the bearings wear out quickly. In an extreme case of a rapidly-spinning massive object, a design flaw or a broken part can cause the entire object to self-destruct if it can't withstand the force required to rotate the angular momentum.

It turns out that the eigenvalues of  $M$  are all positive, and their relative sizes determine the stability of rotation about the eigenvectors. Suppose  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$  are the eigenvectors, that the eigenvalues are  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$ , and that  $\lambda_1 < \lambda_2 < \lambda_3$ . The rotations about  $\mathbf{v}_1$  and  $\mathbf{v}_3$  (those with the extreme eigenvalues) are stable, whereas rotation about  $\mathbf{v}_2$  (with the middle eigenvalue) is unstable. When you start the object spinning about a particular axis, it's impossible to get the axis *exactly* right. If you set the object spinning about an axis that's very close to  $\mathbf{v}_1$  or  $\mathbf{v}_3$ , you won't notice that you're off. The wobbling will be very slight and won't increase. On the other hand, if you try to set it spinning about  $\mathbf{v}_2$  and you are a little off (which you will be with very high probability), the object

will spin a few times without wobbling, but then will begin to wobble. What's happening is that it is really always wobbling. The wobble may be imperceptible at first, but the axis of rotation gets farther from  $\mathbf{v}_2$ , and the wobble then becomes noticeable. Give this a try with your tennis racquet or book. You will find that there are two axes around which it is easy to make it spin smoothly, and that they are perpendicular. If you start it spinning about the direction perpendicular to these two, it will quickly start to wobble.

This has clear implications for something like satellite design. If the satellite has an antenna that's supposed to point in a predictable direction, the satellite needs to spin about a predictable axis. Thus you need at least one of the extreme eigenvalues,  $\lambda_1$  or  $\lambda_3$ , to be very different from the middle one,  $\lambda_2$ . Some early satellites failed spectacularly by tumbling out of control because the designers didn't understand this. They tried to have the satellite rotate about some axis that wasn't an eigenvector, or about some eigenvector whose eigenvalue was too close to the middle eigenvalue. Conversely, many modern, sophisticated satellites use internal robotics to adjust the mass distribution. This changes  $M$  (and its eigenvectors and eigenvalues), allowing for subtle control over the stable axes of rotation.

### CURVATURE OF SURFACES, ASTIGMATISM, AND COMPUTER-AIDED DESIGN OF SMOOTH OBJECTS

Suppose  $S \subset \mathbb{R}^3$  is a surface, for example the hyperbolic paraboloid  $z = x^2 - y^2$ , which is the graph of the function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $f(x, y) = x^2 - y^2$ , or the ellipsoid  $x^2 + y^2/4 + z^2/9$ , which is a level set of the function  $F: \mathbb{R}^3 \rightarrow \mathbb{R}$  given by  $F(x, y, z) = x^2 + y^2/4 + z^2/9$ .

For each point  $p$  on  $S$ , consider a unit vector  $\mathbf{N}$  perpendicular to  $S$  with its tail at  $p$ . As  $p$  moves on  $S$  the vector  $\mathbf{N}$  will, in general, have to change directions due to the surface  $S$  being bent or curved. (The only way for  $\mathbf{N}$  not to change directions at all is for  $S$  to be a plane.) The way  $\mathbf{N}$  changes depends on the direction  $p$  moves and how  $S$  bends.

Suppose you want to design a complicated surface, say a car body. If you are using a CAD (computer-aided design) program, it will have a library of formulas of various surfaces, along with some code that can attach various surfaces together according to your specifications. You would, for example, pick out some surface for the hood of the car, some other surface for the most curved part of the fender, and a third surface for the less curved part of the fender. You would then specify how you wanted the surfaces attached to each other and along what curves. The computer would modify the surfaces, especially near the boundary curves, so that they fit together correctly. (You can probably imagine that this process involves quite a bit of mathematics.) You probably want most of the transitions from one surface to another to be smoothly curved, that is, without edges (like the fold between two faces of a box) or corners—otherwise your car will look like a box! (There may be a few places where you want edges or corners, and you can specify this as well.) To modify two surfaces so they come together without an edge, the computer needs to make the vector  $\mathbf{N}$  continuous. Think about a box again. If a point  $p$  on the box moves from one face to another, the perpendicular unit vector  $\mathbf{N}$  at  $p$  suddenly changes directions as  $p$  crosses the edge— $\mathbf{N}$  is discontinuous along the edge. In fact, it's not even defined on the edge. The discontinuity of  $\mathbf{N}$  is even worse at the corners of the box. Rounding off the

edges and the corners to make a smooth transition is the same as making  $\mathbf{N}$  continuous.

Now if you were going to make the car body out of vinyl, or some other non-reflective substance, this would be enough. But we like cars to be shiny and reflective. If you get close to the car, you can see your reflection and the reflections of other nearby objects. As you or the objects move, the reflections move. The reflections are distorted because the surface of the car is curved, but they aren't torn, and when you move they don't jump suddenly, they move and distort (morph) gradually. For this, the surface needs to be smoother than it would be by simply saying that  $\mathbf{N}$  has to be continuous. We need to have more control over how  $\mathbf{N}$  changes from point to point on the surface.

Of course, quantitatively measuring how something changes involves derivatives. Here's an idea of how this works. Let  $\mathbf{v}$  be a vector tangent to  $S$  with its tail at  $p$ . Such a vector could indicate the velocity (direction and speed) at which  $p$  could move. As  $p$  moves with this velocity, the vector  $\mathbf{N}$  will change at a certain rate. This is a type of derivative of  $\mathbf{N}$ , and it is denoted as  $D_{\mathbf{v}}\mathbf{N}$ . You will study this type of derivative if you take Math 225. There are a number of things to observe about this derivative. First, the notation is meant to indicate that the derivative depends on  $\mathbf{v}$ , the velocity (direction and speed) of  $p$ . If  $p$  moves at a different velocity  $\mathbf{w}$  (again, tangent to  $S$  at  $p$ ), then  $D_{\mathbf{w}}\mathbf{N}$  will, in general, be different from  $D_{\mathbf{v}}\mathbf{N}$ . Since  $\mathbf{N}$  is a vector, its change will also be given by a vector, and the same is true of its rate of change  $D_{\mathbf{v}}\mathbf{N}$ . Furthermore, when the vector  $\mathbf{N}$  changes, it changes in direction only, since its length is constant. As a consequence (this is much less obvious), the derivative  $D_{\mathbf{v}}\mathbf{N}$ , which measures how  $\mathbf{N}$  changes, has to be perpendicular to  $\mathbf{N}$  (if it had a component parallel to  $\mathbf{N}$ , then  $\mathbf{N}$  would be changing length). Thus, if you put the tail of  $D_{\mathbf{v}}\mathbf{N}$  at  $p$ , it will also be tangent to  $S$ . We now have a way to take a vector  $\mathbf{v}$  that's tangent to  $S$  at  $p$  and produce another vector, namely  $D_{\mathbf{v}}\mathbf{N}$ , that is also tangent to  $S$  at  $p$ . We have the makings of a function, we just need to be precise about its domain, range, and the type of function it is. Let  $T_pS$  denote the plane tangent to  $S$  at  $p$ . We consider  $T_pS$  to be a two-dimensional vector space by considering  $p$  to be its origin. In this way, we can then consider  $\mathbf{v}$  and  $D_{\mathbf{v}}\mathbf{N}$  to be elements of this vector space. Thus we have defined a function  $L: T_pS \rightarrow T_pS$  by  $L(\mathbf{v}) = D_{\mathbf{v}}\mathbf{N}$ . Although it's far from obvious, this function is linear. Moreover, it is symmetric, and so it has two orthogonal eigenvectors. The eigenvectors represent the directions  $\mathbf{v}$  at  $p$  in which the surface bends (that is,  $\mathbf{N}$  changes) in the same direction as  $\mathbf{v}$  (as opposed to twisting sideways relative to  $\mathbf{v}$ ). The eigenvalues represent the maximal and minimal amounts of curvature of the surface at  $p$ , and are called the principal curvatures. When the principal curvatures are non-zero and equal at  $p$  (in which case every vector is an eigenvector), the surface can be approximated very well near  $p$  by a sphere. When the principal curvatures are unequal at  $p$ , there are two distinct eigenspaces, and the eigenvectors are called the directions of principal curvature, or simply the directions of curvature. In addition, if the principal curvatures are non-zero, the surface can be approximated very well near  $p$  by an ellipsoid if they have the same sign, and by a hyperboloid if they have opposite signs. The situation is very similar to the eigenvectors and eigenvalues associated with a critical point of a function, but here they occur at every point on the surface  $S$ .

Many people have the visual disorder of astigmatism, which is when the lens of the eye is shaped like part of an ellipsoid as opposed to being spherical. (This is unrelated to

nearsightedness or farsightedness.) Typically astigmatism is reported as two numbers and an angle. The numbers are essentially the principal curvatures of the lens at the center of the pupil. The angle indicates the amount of rotation of the directions of curvature from vertical and horizontal. This condition is easily treated with eyeglasses, but is more difficult to treat with contact lenses. Pure nearsightedness or farsightedness can be treated with a spherical corrective lens, whereas astigmatism must be corrected with an ellipsoidal lens. The axes (directions of curvature) of the corrective lens must line up with those of the eye's lens.

Just as the principal curvatures and directions of curvature affect the optical properties of a lens, they also affect the reflective properties of a shiny surface. If you want reflections of objects to move and distort smoothly as they pass from the hood of the car to the fender, the principal curvatures and the directions of curvature must agree on the transition curve between the surfaces. Getting this to happen is not as hard as it may sound—it is sufficient for all of the second-order derivatives that define the surfaces to agree along the transition curve. However, getting particular reflective properties requires more control over these eigenvectors and eigenvalues, which is somewhat more complicated. CAD programs that do this well are expensive, but worth it if you are serious about this type of design.

## WAVES AND VIBRATIONS

Many physical phenomena consist of vibrations and wave-like behavior: all sounds are produced by vibrations that cause waves in the air, the strings on a stringed musical instrument visibly vibrate, earthquakes cause vibrations and waves inside the earth, light is formed by coupled electric and magnetic waves. The ultimate waves are those of quantum mechanics involving subatomic particles studied in upper-level physics and physical chemistry courses. Waves of these types are represented by functions that are elements of some vector space  $V$  of functions. Generally  $V$  is a space of functions that can be differentiated infinitely many times. Usually the functions have several variables, one of which is time. There will be some linear function  $T: V \rightarrow V$  involving derivatives, usually called a linear differential operator in this context, and the eigenvectors and eigenvalues of  $T$  have physical significance.

An example is the two-dimensional wave operator. Imagine a drumhead, which can be represented by a region  $R$  in the  $xy$ -plane. The vector space  $V$  is the collection of all functions  $f$  of the form  $z = f(x, y, t)$ , where  $(x, y)$  is a point in  $R$ , that is, a point on the drumhead,  $t$  is time, and  $z$  represents the distance the point on the drumhead is from its equilibrium point. In addition, one requires  $f(x, y, t)$  to be 0 when  $(x, y)$  is on the boundary of  $R$ ; the edge of the drumhead is clamped so it doesn't vibrate. (For a vibrating string, the ends of the string are held fixed.) For a fixed  $(x, y)$ , one would expect  $z$  to be a sinusoidal function of  $t$  representing how that point on the drumhead vibrates. The wave operator  $T: V \rightarrow V$  is defined by

$$T(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - k \frac{\partial^2 f}{\partial t^2}.$$

The constant  $k$  depends on the material the drumhead is made of. When the vector space consists of functions, the eigenvectors are called eigenfunctions. The eigenfunctions

represent the fundamental modes of vibration of the drumhead (both their shapes and their resonance properties), and the eigenvalues are related to the frequencies of the fundamental vibrations. Both the eigenfunctions and eigenvalues are closely related to the shape of the drum—drums that are circular, oval, square, and rectangular sound different.

The ultimate linear differential operator is the Schrödinger operator, which is the linear operator associated with Schrödinger's equation. The eigenfunctions of the Schrödinger operator for an atom describe the fundamental states the atom can be in. (This is principally the distribution of the electrons in their orbitals.) The eigenvalues represent the energies associated with the states. The fact that the eigenvalues are discrete implies that the possible energies are not arbitrary, and that to change from one state to another the energy of the atom must “jump” from one energy level to another. To get the atom to do this, a particular quantity of energy must either be added to or removed from (absorbed or emitted by) the atom, which is what gives quantum mechanics its name. All physics majors have to take linear algebra (and even more math), but chemistry majors don't. Essentially all of the mathematics in quantum mechanics is linear algebra on these function spaces, and chemistry majors who have taken linear algebra have a distinct advantage over those who haven't.

ROBERT L. FOOTE, APRIL 2003. UPDATED APRIL 2008