PROPERTIES EQUIVALENT TO INVERTIBILITY

Math 223

We have seen a number of conditions on square matrices that are equivalent to invertibility. Here they all are collected in one place. (Not all are discussed in every textbook.)

Theorem. *Let A be an $n \times n$ matrix. The following statements about A are equivalent (they are all true or all false).*

- *A is invertible, that is, A^{-1} exists.*
- *A is non-singular, that is, the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.*
- *For every $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.*
- *For every $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.*
- *μ_A is one-to-one. Here $\mu_A \colon \mathbb{R}^n \to \mathbb{R}^n$ is the linear map defined by $\mu_A(\mathbf{x}) = A\mathbf{x}$.*
- *μ_A is onto.*
- *Every echelon form for A has n pivots (n leading 1s).*
- *The reduced echelon form for A is I_n, the identity $n \times n$ matrix.*
- *In the LU decomposition for A or PA (where P is some permutation matrix), the matrix U has no zeros on its main diagonal.*
- *A is a product of elementary matrices.*
- *The rows of A are linearly independent.*
- *The columns of A are linearly independent.*
- *The rows of A span $\mathbb{R}^n$.*
- *The columns of A span $\mathbb{R}^n$.*
- *The rows of A form a basis for $\mathbb{R}^n$.*
- *The columns of A form a basis for $\mathbb{R}^n$.*
- *The null space of A, also called the kernel of A, is $\{\mathbf{0}\}$.*
- *The column space of A is $\mathbb{R}^n$.*
- *The row space of A is $\mathbb{R}^n$.*
- $\det A \neq 0$.
- $\operatorname{rank} A = n$.
- *0 is not an eigenvalue of A.*

It should be stressed that for matrices that aren't square, many of these statements don't make sense. For those that do make sense, they aren't necessarily equivalent.

Robert L. Foote, Spring 2003