**Math 14** **Final Exam** **Foote**

3 May 1999 200 Points

*"Show enough work to justify your answers."*

**READ CAREFULLY!** Do **all** parts of Problem 1 (10 points each). Then do any **ten** of the remaining problems (15 points each).

When you use *Mathematica* as an essential part of solving a problem, in order to get full credit you **must** indicate in some detail how you use it, enough so it is clear to me how you draw your conclusions. If in doubt about how much to say, please ask. You do not need to indicate the use of *Mathematica* if you use it simply to check your work. If you have trouble with *Mathematica*, please ask.

Possibly useful formulas: $|I - M_n| \le \dfrac{K_2(b-a)^3}{24n^2}$ $\qquad |I - S_n| \le \dfrac{K_4(b-a)^5}{180n^4}$

1. Warm up problems. Do **all** parts. Do by hand, showing all steps. (10 points each)

   a) Evaluate $\displaystyle\int_0^{\sqrt{e-1}} \frac{x}{x^2+1}\,dx.$

   b) Evaluate $\displaystyle\int_0^{7\pi/12} x\sin(2x)\,dx.$

   c) What is the sum of the following series? Explain.
   $1 - 3 + \dfrac{3^2}{2!} - \dfrac{3^3}{3!} + \dfrac{3^4}{4!} - \dfrac{3^5}{5!} + \ldots$

   d) Let $w = \dfrac{xy^2}{x^2+z}$ and compute $dw$.

   e) Evaluate $\displaystyle\int_0^8 \int_1^{x+1} \frac{\sqrt{x+1}}{\sqrt{y}}\,dy\,dx.$

**READ THIS!!** Do any **ten** of the following problems (15 points each). If you work on more than ten you will get credit for the best ten.

2. Explain why the following integral is improper. Evaluate it by hand, showing all steps.
   $$\int_0^{\pi/2} \tan x\,dx$$

3. Use an appropriate number of terms of the power series for $\sin x$ to give an approximation of $\sin 2$ with an error of less than 0.005. Be clear how many terms you use. Without refering to the approximation of $\sin 2$ given by *Mathematica* or your calculator, carefully explain how you know how many terms to use.

4. Use a Simpson's Rule sum to approximate the value of the following integral with an error less than 0.001. Be clear how you determine the number of intervals to use. You may **not** use *Mathematica*'s approximation of the integral to do this. You may use *Mathematica* to evaluate the approximating Simpson's Rule sum.
   $$\int_0^3 \cos(x^2)\,dx$$

5. The curve $y = e^{-x}$ for $x \ge 0$ is rotated about the $x$-axis. Find the volume of the resulting infinitely long solid.

6. Find the volume under the graph of $z = e^{x^2}e^{y^2}$ over the disk $x^2 + y^2 \le 2$. Do by hand, showing all steps.

7. Explain why the improper integral $\int_1^\infty \frac{1}{x^2 + \ln x}\,dx$ converges. Then find a positive number $b$ such that $\int_b^\infty \frac{1}{x^2 + \ln x}\,dx$ is less than 0.001. Do by hand, showing all steps.

8. Evaluate the following antiderivative by hand, showing all steps. $\int \frac{2 - x^2}{x^3 - x}\,dx$

9. The following picture shows the graph of a function the domain of which is the rectangle in the $xy$-plane $-3 \le x \le 3$, $-3 \le y \le 3$. Based on this graph, draw several level curves of the function. In particular, comment on what the level curves look like near the maximum points and near the saddle point, and how the curves are different around the two maxima. The tops of the hills are at $(1,1)$ and $(-1,-1)$. The saddle point is at $(0,0)$.

10. The following table gives values of a function $f$ for $0 \le x \le 1$, $0 \le y \le .5$. Use these values to give an approximation of $\int_{0.2}^{0.4} \int_{0.3}^{0.7} f(x,y)\,dx\,dy$. Show enough detail so I can tell what you are doing. In particular, draw the region of integration in the $xy$-plane **seperate from the table**, and clearly indicate the subrectangles you are using, their dimensions, and the value of the function you use for each subrectangle.

| $y$ | | | | | | |
|---|---|---|---|---|---|---|
| 0.5 | 0.50 | 0.53 | 0.64 | 0.78 | 0.94 | 1.12 |
| 0.4 | 0.40 | 0.44 | 0.56 | 0.72 | 0.89 | 1.08 |
| 0.3 | 0.30 | 0.36 | 0.50 | 0.67 | 0.85 | 1.04 |
| 0.2 | 0.20 | 0.28 | 0.44 | 0.63 | 0.82 | 1.02 |
| 0.1 | 0.10 | 0.22 | 0.41 | 0.60 | 0.80 | 1.00 |
| 0.0 | 0.00 | 0.20 | 0.40 | 0.60 | 0.80 | 1.00 |
| | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 $x$ |

11. Find three positive numbers $x$, $y$, $z$ such that $x + 2y + 3z = 36$ and $xyz$ is as large as possible. Give some argument indicating how you know you have found the maximum product.

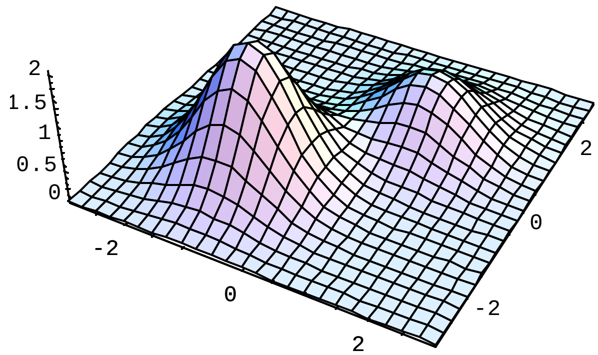
12. Use the method of Lagrange multipliers to find the points on the ellipse $4x^2 + 9y^2 = 36$ that are closest to and farthest from $(2,1)$. You will need to use *Mathematica* to solve the equations. Add the points to the picture and give their coordinates.

13. Consider the graph of $f(x) = x^2$ from $x = 0$ to $x = 3$. Find the point on the curve that is halfway between the endpoints. Give the coordinates accurate to three decimal places.

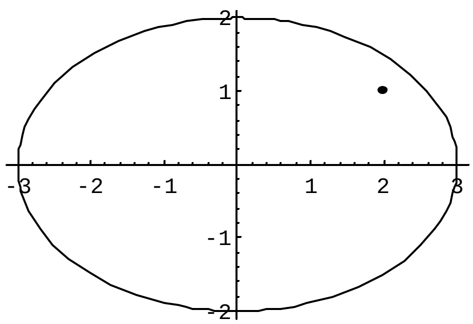
14. Give the first four non-zero terms of the Maclaurin series for $f(x) = \tan x$.

15. Explain why the following series converges. Determine its sum either exactly (you may use *Mathematica*'s value) or give an approximation that has an error of less than 0.01. If you give an approximation, be clear how you know which terms to use. Note: This series does not satisfy the Alternating Series Test!

$$1 - 1 + \frac{1}{3} - \frac{1}{4} + \frac{1}{9} - \frac{1}{9} + \frac{1}{27} - \frac{1}{16} + \frac{1}{81} - \ldots$$




Problem 9




Problem 12

Selected answers and hints.

1. See what *Mathematica* gets (in part c) use it to compute the partial derivatives separately, or use MVC.nb to compute the full differential).
2. See what *Mathematica* gets.
3. You need to use the first four terms (through the seventh power). $\sin 2 \approx 0.907937$
4. $n \geq 36$, $\quad \int_0^3 \cos(x^2)\,dx \approx 0.702898$
5. $\pi/2$
6. $\pi(e^2 - 1)$
7. $b \geq 1000$
8. See what *Mathematica* gets. Note: It will also do the algebra without doing the antiderivative. Use **Apart**[ ... ].

10. There are many correct answers to this. The following uses four subrectangles, each with $\Delta x = .2$ and $\Delta y = .1$. The value used for the function is the average of the values on the upper and lower edges of the subrectangles.
$$\int_{0.2}^{0.4}\int_{0.3}^{0.7} f(x,y)\,dx\,dy \approx \left(\tfrac{.56+.5}{2} + \tfrac{.72+.67}{2} + \tfrac{.5+.44}{2} + \tfrac{.67+.63}{2}\right)(.1)(.2) = .0469$$
11. 288
12. Closest: $(2.245, 1.326)$, farthest: $(-2.95, -.36)$.
13. $(2.054, 4.220)$

15. $3/2 \;-\; \pi^2/6$