Math 14 **Exam 2** **Foote**

17 March 1999 100 Points

"Show enough work to justify your answers."

READ THIS !!! You are to work the problems 1 and 2, and then any four of problems 3 through 8.

When you claim that a series converges or diverges, you must give complete reasons. When using one of the tests, this means giving its name and verifying its hypotheses.

When you use *Mathematica* as an essential part of solving a problem, indicate in some detail how you use it, enough so it is clear to me how you draw your conclusions. If in doubt about how much to say, please ask. You do not need to indicate the use of *Mathematica* if you use it simply to check your work. If you have trouble with *Mathematica*, please ask.

1. What is the sum $\frac{1}{5} + \frac{6}{25} + \frac{36}{125} + \ldots$? (5 points)

2. Let $f(x) = xe^{-x}$. Find the area under the graph of f in the first quadrant. Evaluate the integral you get by hand, showing all steps. (15 points)

   READ THIS !!! Do any **four** of the remaining problems. If you work on more than four, you will get credit for the best four. (20 points each)

3. Consider the improper integral $\int_0^\infty \sqrt{x}e^{-x}\,dx$.
   a) Explain why this integral converges. (Hint: See the previous problem.) (5 points)
   b) Find a value of b so that $\int_0^b \sqrt{x}e^{-x}\,dx$ is within 0.0001 of the value of the above integral. You may **not** use *Mathematica*'s numerical approximation for these integrals. (15 points)

4. Find the interval of convergence, including endpoints, of $\sum_{k=1}^\infty \frac{(x-5)^k}{k2^k}$.

5. Suppose that $a(x)$ is a positive decreasing function and that $a_k = a(k)$, that is, $a(x)$ is the function you would use in the integral test for the series $\sum_{k=1}^\infty a_k$.
   a) Draw a clear, labeled picture illustrating the inequality $\sum_{k=1}^n a_k \geq \int_1^{n+1} a(x)\,dx$. (10 points)
   b) We have seen that harmonic series $\sum_{k=1}^\infty \frac{1}{k}$ diverges, so its sum is ∞. This means that by adding up sufficiently many terms, you can make the sum as big as you like. It's surprising, however, how many terms are needed. Use the inequality in part a) to determine how many terms you need to add up to get a sum that is greater than 10. (10 points)

6. Determine if $\sum_{k=1}^\infty (-1)^k \frac{k}{k^2+5}$ converges absolutely, converges conditionally, or diverges.

7. Determine if $\sum_{k=1}^\infty (-1)^k \frac{k}{k^3+5}$ converges absolutely, converges conditionally, or diverges.

8. The point of this problem is to approximate $\int_0^2 \frac{\sin x}{x}\,dx$ with a series.
   a) Use the power series for $\sin x$ to get a power series for $\frac{\sin x}{x}$. (5 points)
   b) Integrate the series in part a) to get a series of constants that equals the integral. (7 points)
   c) Approximate the sum of the series with an error less than .01 by adding up an appropriate number of terms. Be sure it's clear which terms you are using, and why you know that the error is less than .01. You may **not** use *Mathematica*'s approximation of the sum of the entire series. (8 points)

Selected answers and hints.

1. ∞
2. See what *Mathematica* gets.
3. a) Note that $\sqrt{x} < x$ for $x > 1$ and compare with the integral in the previous problem.
   b) $b = 11.8$ or larger (it doesn't have to be an integer).
4. $3 \leq x < 7$ or $[3, 7)$
5. b) 22026 terms
6. This converges conditionally, which involves showing *two* things, that the given series converges and that a related series diverges.
7. This converges absolutely, which involves showing *one* thing, that a related series converges.
8. c) You need to use the first three non-zero terms. Their sum is 1.60889. The error made by this is actually less than .0037, but this observation was not required.