**Math 14** **Final Exam** **Name:** ______________________________

13 December 1999 100 Points

*"Show enough work to justify your answers."*

**READ THIS !!!** Do any **ten** problems. If you work on more than ten, you will get credit for the best ten. (10 points each)

When you use *Mathematica* as an essential part of solving a problem, indicate in some detail how you use it, enough so it is clear to me how you draw your conclusions. It's not enough simply to say that you used it! If in doubt about how much to say, please ask. You do not need to indicate the use of *Mathematica* if you use it simply to check your work. If you have trouble with it, please ask.

$$|I - M_n| \le \frac{K_2(b-a)^3}{24n^2}, \qquad |I - S_n| \le \frac{K_4(b-a)^5}{180n^4}$$

1. Write an essay discussing the significance of the two formulas above. You should include definitions of the notation, what they are used for, how they are interpreted, and a comparison of them with each other. You do **not** need to illustrate their use.
2. Give an approximation of the following integral with an error less than $10^{-4}$. Be sure to explain how you know your approximation is within the desired accuracy. You may **not** use *Mathematica*'s approximation for the integral, however you **may** use *Mathematica* to evaluate an approximating sum for the integral and to do arithmetic.
$$\int_0^\pi \cos(\cos x)\,dx$$
3. Evaluate by hand, showing all steps: $\int e^x \cos x\,dx$
4. Evaluate by hand, showing all steps: $\lim_{x\to 0^+} (1+2x)^{1/x}$
5. A contour plot of $f(x,y) = \sin x \cos y$ is shown. Note that the origin is not in the center of the plot. Mark all maxima, minima, and saddle points on the plot, and indicate how you know. This does not require any computation, but you may do some if it helps your explaination.
6. Show that the following series converges and approximate its value with an error less than $10^{-4}$. Be sure to explain how you know your approximation is within the desired accuracy. You may **not** use *Mathematica*'s approximation for the sum, however you **may** use *Mathematica* to evaluate a partial sum.
$$\sum_{k=1}^{\infty} \frac{1}{2^k\sqrt{k}}$$
7. Explain why the following series does not satisfy the alternating series test. Show that it converges. Use *Mathematica* to find its exact value. If you don't see the pattern, please ask.
$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + \frac{1}{9} - \frac{1}{8} + \frac{1}{16} - \frac{1}{16} + \frac{1}{25} - \frac{1}{32} + \frac{1}{36} - \dots$$
8. Let $f(x) = \ln(1+x)$.
   a) Compute the Maclaurin series for $f$.
   b) What is the interval of convergence for this series?
   c) Use the series to show that $\ln 2$ is the sum of the alternating harmonic series.

9. A five-foot chain that weighs ten pounds is coiled up on the ground under a window that is twelve feet above the ground. Compute the work done in pulling the chain up by a string until the top link is at the window ledge and the rest of the chain is hanging vertically below the window.

10. Find the maximum and minimum values of $f(x, y) = 3x^2 + 2xy + 4y^2 + 2x - 3y$ and their locations on the square given by $-1 \le x \le 1$, $-1 \le y \le 1$.

11. The following table gives selected values of a function $f(x, y)$ on the square $0 \le x \le 1$, $0 \le y \le 1$. Use this data to approximate $\int_0^1 \int_x^1 f(x, y)\, dy\, dx$. Write enough so I can tell what you are thinking.

| $y$ | | | | | |
|---|---|---|---|---|---|
| .9 | .18 | .15 | .13 | .11 | .10 |
| .7 | .54 | .45 | .38 | .33 | .30 |
| .5 | .91 | .75 | .62 | .54 | .50 |
| .3 | 1.3 | 1.0 | .88 | .76 | .71 |
| .1 | 1.6 | 1.3 | 1.1 | .98 | .91 |
| | .1 | .3 | .5 | .7 | .9 $x$ |

12. Here's an example illustrating the bizarre world of the infinitely long. The graphs of $y = 1/x$ and $y = -1/x$ are pictured on the interval $x \ge 1$.
    a) Show that the area contained between them is infinite.
    b) On the other hand, show that the volume contained by rotating them around the $x$-axis is finite!

13. Evaluate the following integral, in which $R$ is the region in the first quadrant outside the circle $x^2 + y^2 = 1$ and below the line $x + y = 2$. Do by hand, showing all steps.
$$\iint_R \frac{1}{(x^2 + y^2)^{3/2}}\, dA$$

Answers and hints for selected problems.

2. $\int_0^\pi \cos(\cos x)\,dx \approx M_{114} \approx S_{18} \approx 2.40394$
4. $e^2$
6. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k\sqrt{k}} \approx \sum_{k=1}^{14} \frac{1}{2^k\sqrt{k}} \approx .806111$
7. $\dfrac{\pi^2}{6} - 1$
8. a) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \dots$

   b) $-1 < x \leq 1$. Note: if you do part a) by integrating a power series, you only need to check the endpoints.
9. 95 foot-pounds
10. There are nine points that need to be checked: one interior critical point, one point on each of the sides of the square, and the corners of the square.
    Maximum value of 10 occurs at $(1, -1)$ and $(-1, -1)$.
    Minimum value of $-5/4$ occurs at $(-1/2, 1/2)$.
11. Two possible answers:
    $(.2)^2(.18+.15+.13+.11+.10+.54+.45+.38+.33+.91+.75+.62+1.3+1.0+1.6) = .342$
    $(.2)^2\left(.18+.15+.13+.11+.54+.45+.38+.91+.75+1.3+\frac{.10+.33+.62+1.0+1.6}{2}\right) = .269$
12. Both are given by improper integrals on the interval $[1, \infty)$. The volume is $\pi$.
13. $\dfrac{\pi}{2} - 1$