Math 14 **Exam 2** **Name:** ______________________________

1 November 1999 100 Points

"Show enough work to justify your answers."

READ THIS !!! You are to work the problems 1, 2, and 3, and then any four of problems 4 through 9.

When you claim that a series converges or diverges, you must give complete reasons. When using one of the tests, this means giving its name and verifying its hypotheses.

When you use *Mathematica* as an essential part of solving a problem, indicate in some detail how you use it, enough so it is clear to me how you draw your conclusions. If in doubt about how much to say, please ask. You do not need to indicate the use of *Mathematica* if you use it simply to check your work. If you have trouble with *Mathematica*, please ask.

1. What is the exact value of the sum $\frac{\pi}{6} - \frac{\pi^3}{6^3 3!} + \frac{\pi^5}{6^5 5!} - \frac{\pi^7}{6^7 7!} + \ldots$ and why? (5 points)

2. Evaluate the following limit in which n is a fixed (relative to x) positive integer (do more than simply give its value). We have often said that "exponentials grow much faster than polynomials." Explain why this limit justifies this intuitive statement. (7 points)
$\lim\limits_{x \to \infty} \frac{x^n}{e^x}$

3. Give an example of a *series* other than the harmonic series that diverges, for which the corresponding *sequence* of terms converges to 0. Briefly explain why the series diverges and the sequence converges. (8 points).

READ THIS !!! Do any **four** of the remaining problems. If you work on more than four, you will get credit for the best four. (20 points each)

4. Consider the improper integral $\int_0^\infty \frac{1}{e^x + \sqrt{x}}\,dx$.
   a) Explain why this integral converges. (5 points)
   b) Find a value of b so that $\int_0^b \frac{1}{e^x + \sqrt{x}}\,dx$ is within 0.0001 of the value of the above integral. You may **not** use *Mathematica*'s numerical approximation for these integrals. (15 points)

5. Suppose that f is a function that satisfies $f(0) = 1$, $f'(0) = 0$, and $f''(x) = 2f(x)$. Find the Maclaurin series for f. Give enough terms so that the pattern is clear. Hint: Use the last equation to find $f''(0)$. Then differentiate it to get information about the higher-order derivatives.

6. Find the interval of convergence, including endpoints, of $\sum_{k=1}^{\infty} \frac{(x+5)^k}{k^2 3^k}$.

7. Determine if $\sum_{k=1}^{\infty} (-1)^k \frac{k^2}{k^3 + 10}$ converges absolutely, converges conditionally, or diverges.

8. Determine if $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{3^k + 10}$ converges absolutely, converges conditionally, or diverges.

9. The point of this problem is to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ with a series.

   a) Use the power series for e^t to get a power series for e^{-x^2}. (5 points)
   b) Use the series in part a) to get a series of constants that equals the integral. (7 points)
   c) Approximate the sum of the series with an error less than 0.001 by adding up an appropriate number of terms. Be sure it's clear which terms you are using, and why you know that the error is less than 0.001. You may **not** use *Mathematica*'s approximation of the sum of the entire series. (8 points)

Answers and hints to selected problems.

1. 1/2

4. b) An easily-obtained lower bound for b is $4\ln 10$. Any larger value will work, for example b could be 10. It doesn't have to be an integer.

5. $1+\dfrac{2}{2!}x^2+\dfrac{4}{4!}x^4+\dfrac{8}{6!}x^6+\dfrac{16}{8!}x^8+\dfrac{32}{10!}x^{10}+\dots$

6. $-8\le x\le -2$. In this example you can get convergence at both endpoints by noting that the one with positive terms converges.

7. The series converges conditionally. Note that this requires proving that the given series converges and that a related series diverges. You have to be careful when using the Alternating Series Test, since the terms don't decrease at first!

8. The series converges absolutely. This can be shown by the comparison test or by the ratio test, but not applied to the original series!

9. Approximation of $\int_0^1 e^{-x^2}\,dx$ with a series.

   b) $\int_0^1 e^{-x^2}\,dx = 1-\dfrac{1}{3}+\dfrac{1}{5\cdot 2!}-\dfrac{1}{7\cdot 3!}+\dfrac{1}{9\cdot 4!}-\dfrac{1}{11\cdot 5!}+\dots$

   c) $\int_0^1 e^{-x^2}\,dx \approx 1-\dfrac{1}{3}+\dfrac{1}{5\cdot 2!}-\dfrac{1}{7\cdot 3!}+\dfrac{1}{9\cdot 4!}\approx .74749$