Math 14 **Exam 1** **Name:** ______________________________

24 September 1999 100 Points

"Show enough work to justify your answers."

1. Evaluate the following antiderivatives. Show all steps—don't just give answers.
   a) $\int \sin\sqrt{x}\,dx$. Use the substitution $u^2 = x$. (13 points)

   b) $\int \frac{e^x}{e^{2x}+1}\,dx$ (12 points)

   c) $\int \frac{x+1}{(x-1)(x^2+1)}\,dx$ (13 points)

   d) $\int \sin^3 x\,dx$. (12 points)

$$|I - T_n| \le \frac{K_2(b-a)^3}{12n^2}, \qquad |I - M_n| \le \frac{K_2(b-a)^3}{24n^2}, \qquad |I - S_n| \le \frac{K_4(b-a)^5}{180n^4}$$

2. Consider the formulas above. (5 points each)
   a) Why do these formulas say something about the error made by sums that approximate an integral? Why do they say that using a large number of subintervals is generally better? You may use any of the three formulas to illustrate your comments.
   b) Refering to the formulas, explain why we generally expect Simpson's rule to do a better job approximating an integral than the trapezoid or midpoint sums.

3. You may use *Mathematica* in this problem. You should indicate in some detail how you use it, enough so it is clear to me how you draw your conclusions. (Don't simply say that you used *Mathematica*.) If in doubt about how much to say or if you have trouble with *Mathematica*, please ask. You may **not** use *Mathematica*'s approximation for the integral, however you **may** use *Mathematica* to evaluate an approximating sum for the integral and to do arithmetic.

   Suppose you want to approximate the integral $\int_0^{10} \frac{x}{x+\cos x}\,dx$ with an error less than .001.
   a) Determine how many subintervals are needed to do this for a midpoint sum, a trapezoid sum, or a Simpson's rule sum (choose one). Be clear which type of sum you are considering, and how you know the number of subintervals is adequate. (15 points)
   b) Based on part (a), what is your approximation for the integral? (5 points)

4. The following table of data gives some values of a function $g(x)$.

x	3.	3.5	4.	4.5	5.
$g(x)$	3.56	3.78	3.85	3.73	3.42

   a) Use these data to compute T_4 for $\int_3^5 g(x)\,dx$. You may use your calculator or *Mathematica* to do the arithmetic, but write enough so I can tell how you are using the data. (15 points)
   b) Would you expect T_4 to overestimate or underestimate the true value of the integral? Explain. (5 points)

5. **Extra Credit.** Do **one** of the following (use the back of this page). If you work on both, you will get credit for the best one. (10 points)
   a) Show that the error made by approximating $\int_0^2 x^2\,dx$ with a trapezoid sum with one interval is the maximum possible allowed by the trapezoid error estimate.
   b) If f is a linear function (so its graph is a line), then any trapezoid sum for $\int_a^b f(x)\,dx$ will be exact. This is because the tops of the trapezoids will lie exactly on the graph of f and so the error will be zero. Similarly, since Simpson's Rule is based on approximating the graph of f with parabolas, if f is quadratic (so its graph is a parabola), then any Simpson's Rule sum for $\int_a^b f(x)\,dx$ will be exact. Surprisingly, Simpson's Rule does even better: if f is any *cubic* polynomial, then Simpson's Rule is exact! Prove this. More specifically, suppose that $f(x) = Ax^3 + Bx^2 + Cx + D$, and prove that the error made by a Simpson's Rule sum is zero.

Answers and hints to selected questions.

1. See what *Mathematica* gets.
3. Numerical approximation of integral.
   a) Trapezoid: $n \geq 409$. Midpoint: $n \geq 289$. Simpson's: $n \geq 72$.
   b) $\int_0^{10} \frac{x}{x+\cos x}\,dx \approx 10.0699$
4. Trapezoid sum from data.
   a) $T_4 = 7.425$
   b) Make a judgement based on a plot the data.