Math 13 **Final Exam** **Foote**

15 December 1998 200 Points

"Show enough work to justify your answers."

Part I. Do **all six** of the problems in this part. Show all steps of every computation. Your answers must be independent of *Mathematica*. (10 points each)

1. Let $f(x) = x(|x| + 1)$. Compute $f'(0)$ using the definition of derivative.
2. Evaluate $\lim_{x \to 3} \dfrac{x^2 - x - 6}{x^2 - 3x}$.
3. Evaluate $\int 2^x \sqrt{1 + 2^x}\, dx$ using the substitution $u = 1 + 2^x$.
4. Evaluate $\int_0^{\pi/6} \sin x \cos x \, dx$ using a substitution.
5. See what *Mathematica* gets for an antiderivative of $\sqrt{1 - x^2}$ and verify by hand that it is correct.
6. Find the equation of the line tangent to the graph of $8xy^2 = (x + y)^4$ at the point $(1/2, 1/2)$.

Part II. Do any **seven** of the problems in this part. If you do more than seven, you will get credit for the best seven. Note that some problems have multiple parts. If you use *Mathematica*, be very clear what you use it for, and show enough work that I can tell what you are thinking. If you have any question about what constitutes sufficient work, be sure to ask. (20 points each)

7. The graph of a function f is shown (below). Suppose $F(x) = \int_0^x f(t)\, dt$. Answer the following. (5 points each)
   a) Where is F increasing? Why?
   b) Explain why $F''(3) = 0$ but F does *not* have an inflection point at $x = 3$.
   c) What is an equation of the line tangent to the graph of F at $x = 1$?
   d) Does the graph of F lie above or below its tangent line at $x = 1$? How do you know?
8. We have talked about antiderivatives and integrals in this course. Write an essay discussing the difference between these two concepts, the theorem that relates them, and how they are related.
9. A box without a lid is to be constructed from an 8in by 8in square piece of cardboard by cutting small squares out of the corners and folding up the sides. The cutouts must be at least 1in by 1in. What are the largest and smallest possible volumes for such a box, and how big should the cutouts be to get these volumes?
10. Let $F(x) = \int_1^x \sqrt{1 + t^3}\, dt$. Find the quadratic approximation to F at $x = 1$ and use it to approximate $F(1.2)$.
11. The following table gives values of a function f for some values of its input.

x	0	0.5	1.	1.5	2
$f(x)$	0	1.875	3	2.625	0

   a) Use a trapezoid sum to approximate $\int_0^2 f(x)\, dx$. Show enough so I can tell what you are doing. (15 points)
   b) Do you think that this approximation overestimates or underestimates the true value of the integral? Explain. (5 points)

12. Let f be the function in the previous problem. Assuming f' and f'' exist, find approximations for $f'(1)$ and $f''(1)$. Explain or make your computations clear enough that I can tell how you are computing.

13. Use Newton's Method to approximate a value of x for which $e^{-x} = x/2$. Clearly indicate the formula you are using, in simplified form, your initial guess, and all subsequent approximations until they start repeating to six decimal places.

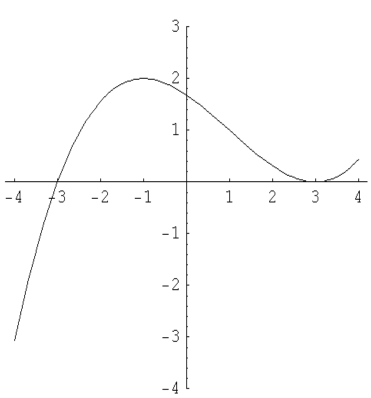
14. The pictured curve (below) is $y = 4x - x^3$. Find the value of c so that the area under the curve in the first quadrant to the left of $x = c$ is half of the area under the whole curve in the first quadrant. To find c you will need the *Mathematica* **Solve** command. Give both exact and decimal approximations for c.

15. Find the area bounded by the curve $y = x^3 - 2x^2 + 2$ and the line $y = x$.

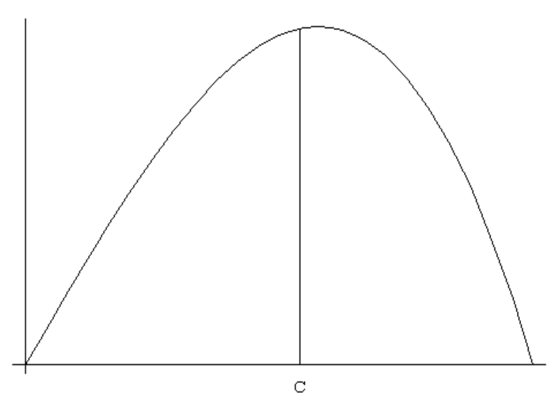
16. A bicyclist is traveling at 20 ft/sec east on County Road A at the same time a car is traveling at 80 ft/sec north on County Road B. At the same moment, the bicyclist is 30 feet east of the intersection of the two roads, and the car is 40 feet south of the intersection. At this moment, are they getting closer or farther from each other? How fast?

17. The differential equation that expresses Newton's Law of Cooling is $\frac{dT}{dt} = k(T - C)$ in which T is the temperature of a body, say a cup of coffee, as a function of time, C is the temperature of the air (assumed to be constant), and k is a constant that measures thermal conductivity (the ability of heat to pass from the cup to the air).
    a) Show that $T = C + Ae^{kt}$ is a solution of the differential equation. (5 points)
    b) Suppose that the coffee is initially 210°F and that after ten minutes it is 190°F. Assuming the temperature of the room is 70°F, find a simplified formula for T as a function of time. (Remember, it's easier if you don't find the value of k. Instead, find the value of e^k.) (10 points)
    c) At what time will the coffee be 170°F? (5 points)

Part III. Extra Credit. This may not be used as one of the seven problems for Part II. In the problem about the bicycle and the car, how fast is the distance between the car and the bicycle changing when they are closest to each other? You do not need to have done that problem to do this. Explain. (10 points)



Problem 1



Problem 8