Math 13 **Final Exam** **Foote**

18 December 1997 200 Points

"Show enough work to justify your answers."

Read This!

- You may use *Mathematica* or a graphing calculator. When their use is significant in the solution of a problem, you should indicate this, and include the relevant output (formulas or graphs) on your exam. If you need help with *Mathematica* or unsure of what you need to write down, please ask.
- If you are sharing the computer, give the other person a chance to use it! When you are done using the computer, erase your work and see if the other person needs it.
- Exact answers are expected, unless otherwise indicated.

1. Find the values of the following, showing **all** computations and reasoning (answers must be independent of *Mathematica*). (10 points each)

   a) $\lim_{x\to 2} \dfrac{2x^2-3x-2}{x^2+x-6}$ b) $\displaystyle\int_0^{\sqrt{\pi/3}} x\cos x^2\,dx$ c) $\displaystyle\int_{-4}^{0} \sqrt{16-x^2}\,dx$

2. Show **all** computations and reasoning used in the following (answers must be independent of *Mathematica*). (10 points each)

   a) Let $f(x)=\ln(x^2+1)$. Find the tangent line at $x=2$.
   b) Evaluate $\int x^2\sqrt{x+3}\,dx$ using the substitution $u=x+3$.
   c) Show that $\int \ln x\,dx = x\ln x - x + C$.

   READ CAREFULLY! Twelve problems follow. Work on at least seven of them. If you work on more than seven, I will use the best seven to determine your exam score. Note that some of the problems have multiple parts. (20 points each)

3. Sketch the graph of a function defined for $0 \quad x \quad 6$, $x \neq 1$ with the following properties. You may want to make some rough sketches on scratch paper before deciding on your final graph. (10 points)

   f is continuous except at $x=1$,
   $f(0)=f(4)=f(6)=1$, $f(3)=0$, $\lim_{x\to 1} f(x) = -\infty$,
   $f'(0)=f'(5)=0$, $f'(3)=-1$, $f'(4)$ is undefined,
   $f''(x)<0$ for $0 \quad x<1$ and $1<x<3$, $f''(x)>0$ for $3<x<4$ and $4<x<6$.

4. The average of 3 and x is $f(x)=\frac{3+x}{2}$. This is also known as their *arithmetic mean*. Another way to average them is their *geometric mean*, which is $g(x)=\sqrt{3x}$. Determine, **with proof**, which is larger for $x>0$, $f(x)$ or $g(x)$. Note that they are equal when $x=3$. One may be larger for some values of x and the other for different values of x. Be sure to consider all possibilities. Remember, to prove inequalities we use the Racetrack Principle and its variations. It may help to see how f and g are related near the place where they are equal.

5. The two curves pictured (below) are $f(x)=\cos x$ and $g(x)=1-\frac{x^2}{2}$ (you need to determine which is which). Compute the amount of shaded area. Give the exact answer and a numerical approximation.

6. The equation $4xy = (x+y)^4$ defines y as a function of x, $y = f(x)$, near the point $(1/2, 1/2)$ (see picture).
   a) Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at this point. (10 points)
   b) What is the quadratic approximation for f near $x = 1/2$? (5 points)
   c) Use the quadratic approximation to determine the approximate value of y such that $(.6, y)$ is on the graph near $(1/2, 1/2)$. (5 points)

7. The following table gives values for a function f.

x	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.00	.90	.75	.55	.30	.00

   a) Find an approximation for $f'(.5)$. Do your work above. (8 points)
   b) Find an approximation for $\int_0^1 f(x)\,dx$. Specify what type of sum you are using (that is, right sum, left sum, trapezoid sum, or midpoint sum). (8 points)
   c) Is the approximation you get an under estimate or over estimate? Explain. (4 points)

8. A bug is crawling up a wall at 2 inches per second. A cat is stalking the bug and is moving towards the wall at 6 inches per second. At a particular moment the bug is 4 feet above the floor and the cat is 3 feet from the wall. Determine if the cat is getting closer to the bug or farther from it at that moment.

9. Evaluate $\lim_{x\to\infty} \dfrac{\sqrt{x^2 - x + 5}}{2x + 7}$, showing all steps (answer must not depend on *Mathematica*).

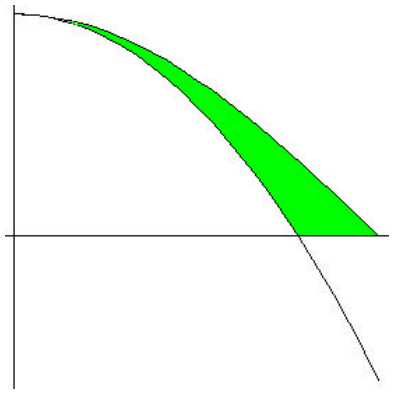
10. The derivative of a function F is shown below.
    a) On what intervals is F increasing? How do you know?
    b) On what intervals is F concave up? How do you know?
    c) Assuming that $F(-1) = 3$, what is the equation of the line tangent to the graph of F at $x = 2$?

11. Consider the equation $x = \cos x$.
    a) Explain why the equation must have a solution. (What theorem guarantees this and why?) (5 points)
    b) Illustrate the approximation of the solution with Newton's Method. Start with an integer close to the solution and use two interations. (15 points)

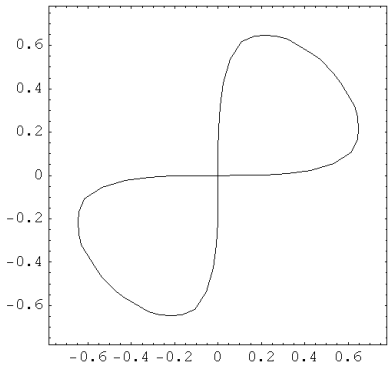
12. Evaluate $\displaystyle\int \frac{e^x}{1 + e^{2x}}\,dx$, showing all steps (answer must not depend on *Mathematica*).

13. Find values of a, and b so that $f(x) = a\sin bx$ satisfies $f''(x) = -9f(x)$ and $f'(\pi/3) = -6$.

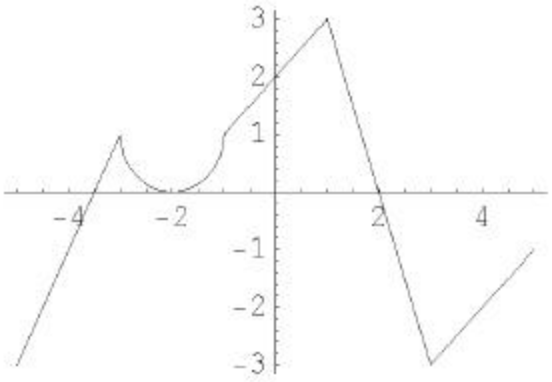
14. A rocket is fired from an initial altitude of 1000 meters. After t seconds its altitude is given by $f(t) = -t^4 + 18t^3 + 1000$.
    a) Make a sketch of the altitude of the rocket from the time it is launched until the time it crashes (when it's altitude is zero). Include scales and units on the axes. Make a rough estimate (to the nearest second) of when it crashes. (6 points)
    b) How high does the rocket go, and when does it reach this altitude? Explain. (8 points)
    c) What is the maximum upwards velocity of the rocket, and when does this happen? Explain. (8 points)



Problem 5



Problem 6



Problem 10

Have a Good Holiday!