Math 112 **Final Exam** **Name:**

30 April 2007 200 Points

"Show enough work to justify your answers."

READ THIS !! You may use *Mathematica* on any problem to help you think. You may use it as part of your solution only on indicated problems.

READ THIS !! Exact answers are expected except for problems involving approximations, in which case you may use decimal approximations.

1. Do **all six** parts of this problem, showing all work on these pages. (10 points each)

   (a) Evaluate $\displaystyle\int_0^{2\pi/3} \sin x\sqrt{3+\cos x}\,dx$.

   (b) What is the sum of the following series?
   $2-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{2}{9}+\dfrac{4}{27}-\dfrac{8}{81}+\dots$

   (c) Find an equation of the plane tangent to the graph of $xz^2+x^2y-z^3=13$ at the point $(2,3,1)$. Leave the equation in a form that exhibits the point of tangency.

   (d) The following table gives values of a function f. Use this to give the T_5 approximation of $\int_0^1 f(x)\,dx$. Show enough work so I can tell what you are doing.

x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$f(x)$	1	0.96	0.85	0.68	0.49	0.29	0.13	0.03	0.00	0.05	0.17

   (e) Evaluate $\displaystyle\int \sec^4 x\,dx$.

   (f) Give the power series expansion about 0 of the function $\cos^2 x = \frac{1}{2}(1+\cos 2x)$. Give at least six non-zero terms.

READ THIS !! Do any **seven** of the remaining problems. If you work on more than seven, you will get credit for the best seven. (20 points each) Please work each problem on a separate sheet of paper. Put the problem number in the upper right corner. Avoid writing in the upper left corner where the staple will go.

READ THIS !! You may use *Mathematica* as part of your reasoning only where indicated. If you use *Mathematica* as part of your reasoning on a problem, indicate that on the problem. Save your *Mathematica* work in a single file with the problems in order. When you are done with the exam, print a copy of the file *and* e-mail it to me. If you have trouble with *Mathematica*, please ask.

$$|I - T_n| \leq \frac{K_2(b-a)^3}{12n^2} \qquad\qquad |I - M_n| \leq \frac{K_2(b-a)^3}{24n^2}$$

2. Suppose that f is a function that is concave up and decreasing on $[a, b]$. Let $I = \int_a^b f(x)\,dx$.

   (a) Put the following in increasing order: L_{10}, R_{10}, T_{10}, M_{10}, I. Write your answer as a sequence of inequalities, that is, in the form $A \leq B \leq C \leq D \leq E$. (5 points)

   (b) Explain why each of the four inequalities is true. (15 points)

3. Suppose you want to approximate the integral $\displaystyle\int_0^5 \frac{x+1}{x+e^{-x}}\,dx$ with an error less than .001. You may use *Mathematica*.

   (a) Use an error estimate formula to determine how many subintervals are needed to do this for a trapezoid sum or a midpoint sum (choose one). Be clear which type of sum you are considering. Include a proof of how you know the number of subintervals is adequate. You may not use *Mathematica*'s approximation for the integral in your answer. (15 points)

   (b) Based on part (a), what is your approximation for the integral? (5 points)

4. Laplace's equation in two variables is $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Determine if $f(x, y) = \ln(x^2 + y^2)$ satisfies Laplace's equation.

5. Evaluate the following by hand by converting it to polar coordinates, showing all steps.
   $$\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{1}{x^2+y^2+1}\,dy\,dx$$

6. A ball that is shot vertically from the floor reaches a height of 9 feet. Each time it bounces it comes up to 2/3 the height of the previous bounce. Use a series to determine how far does the ball goes. The distance traveled between consecutive bounces can be one term of the series.

7. Evaluate $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{2x^2+4x+5}}$.

8. Explain why $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ is improper. Determine if it converges, and if it does, its value.

9. Determine, with proof, if $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{k^2}{2^k}$ converges absolutely, converges conditionally, or diverges. To get full credit you must indicate which tests you use, and give full details of their use.

10. A rectangular box has a volume of 20 cubic feet. The material for the sides, bottom, and top cost \$1, \$2, and \$3 per square foot, respectively. Find the dimensions and cost of the most economical box.

11. The region between $y = x^2$ and $y = 2x$ is rotated around the y-axis. Find the volume of the resulting solid.

12. Let $f(x) = \sec x$. Find a point $(c, f(c))$ so that the length of the graph of f from $(0, 1)$ to $(c, f(c))$ is within 1 of 100. You may use *Mathematica* to evaluate the appropriate integral, and you may use "guess and check." Before you begin, take a moment to think about what the graph looks like. *Mathematica* has a rough time with the $\int_a^b F(x)\,dx//N$ form on this one. You need to use **NIntegrate[F[x], {x, a, b}]** instead.

Selected answers and hints.

1. (a) See what *Mathematica* gets.
   (b) What kind of series is it starting with the second term?
   (d) .411
   (e) See what *Mathematica* gets.
   (f) $1 - \dfrac{2x^2}{2!} + \dfrac{2^3x^4}{4!} - \dfrac{2^5x^6}{6!} + \dfrac{2^7x^8}{8!} - \dots$
2. (a) $R_{10} \leq M_{10} \leq I \leq T_{10} \leq L_{10}$
   (b) It is not adequate to say that M_{10} is closer to I than R_{10} because the midpoint sum is generally more accurate. It is *not* always more accurate. You need a specific reason why R_{10} is less than M_{10}.
3. (a) For a midpoint sum you need 83 subintervals.
   (b) 6.61094
5. See what *Mathematica* gets.
6. 54 feet
7. Note that the answer can't be positive.
8. Diverges.
9. Using the AST is a waste of time with this problem! Why?
10. \$60
11. The point $(1.5609, 101.049)$ is on the curve at a distance of 100.643 from $(0, 1)$ along the curve.