Math 112 **Final Exam** **Name:**

2 May 2006 200 Points

"Show enough work to justify your answers."

READ CAREFULLY! Do **all** parts of Problem 1 (10 points each). Then do any **seven** of the remaining problems (20 points each).

Possibly useful formulas: $|I - T_n| \leq \frac{K_2(b-a)^3}{12n^2}$ $|I - M_n| \leq \frac{K_2(b-a)^3}{24n^2}$ $|I - S_n| \leq \frac{K_4(b-a)^5}{180n^4}$

1. Warm up problems. Do **all six** parts. Do by hand in the space provided, showing all steps. (10 points each)

   (a) Evaluate $\dfrac{\partial^3}{\partial x^2 \partial y}\left(x^3 e^{5y} + \dfrac{\sin 2x}{y+1}\right)$.

   (b) Give the power series expansion about x of the function $\dfrac{\cos x - 1}{x^2}$.

   (c) Evaluate $\lim\limits_{x \to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$.

   (d) Evaluate $\displaystyle\int \cos^3 x \, dx$

   (e) What is the sum of the following series? Explain.
   $\dfrac{1}{3} + \dfrac{5}{12} + \dfrac{25}{48} + \dfrac{125}{192} + \ldots$

   (f) Evaluate $\displaystyle\int_0^{\sqrt{7}} \frac{x}{\sqrt{x^2+2}}\, dx$

READ THIS!! Do any **seven** of the remaining problems (20 points each). If you work on more than seven you will get credit for the best seven.

When you use *Mathematica* as an essential part of solving a problem, in order to get full credit you **must** indicate in some detail how you use it, enough so it is clear to me how you draw your conclusions. If in doubt about how much to say, please ask. You do not need to indicate the use of *Mathematica* if you use it simply to check your work. If you have trouble with *Mathematica*, please ask.

2. Suppose you want to approximate the integral $\displaystyle\int_0^{10} \frac{x}{x+\cos x}\,dx$ with an error less than .001.

   (a) Determine how many subintervals are needed to do this for a midpoint sum, a trapezoid sum, or a Simpson's rule sum (choose one). Be clear which type of sum you are considering. Include a proof of how you know the number of subintervals is adequate. (15 points)

   (b) Based on part (a), what is your approximation for the integral? (5 points)

3. The finite region between the curves $y = \sqrt{x}$ and $y = x/2$ is rotated around the x-axis. Find the volume generated. Note that the cross sections perpendicular to the x-axis are annuli (the area between two circles).

4. Evaluate the following by hand, showing all steps, by converting it to polar coordinates.
   $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

5. Compute the beginning of the power series for $f(x) = \sec x$ expanded about $x = 0$. Give the first six non-zero terms.

6. Find, with proof, a number b such that $\displaystyle\int_b^{\infty} \frac{1}{\sqrt{x^3+1}}\,dx < 0.005$, or prove that such a number does not exist.

7. Evaluate $\displaystyle\lim_{x\to\infty} x^{1/x}$. Do all work by hand.

8. Explain why the following integral is improper. Then evaluate it by hand showing all steps.
   $\displaystyle\int_0^{\infty} xe^{-x}\,dx$

9. The area of a triangle is given by $A = \frac{1}{2}xy\sin\theta$, where x and y are lengths of two of the sides and θ is the included angle. For a particular triangle $x = 5$, $y = 8$, and $\theta = 60°$. Suppose x increases by $1/5$ and y decreases by $1/10$. Use differentials to find an approximation for how much θ should change so that the area of the triangle doesn't change. Be sure to indicate if θ increases or decreases and if your answer is in degrees or radians.

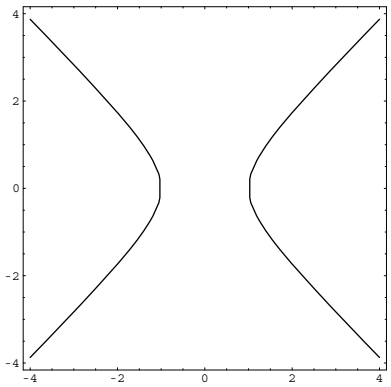
10. Determine with proof if $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{k^2}{k^3+10}$ converges absolutely, converges conditionally, or diverges.

11. The hyperbola $x^2 - y^2 = 1$ is pictured. Use Lagrange multipliers to find the point on the hyperbola closest to $(1, 3)$. The function you minimize should be the square of the distance to $(1, 3)$.

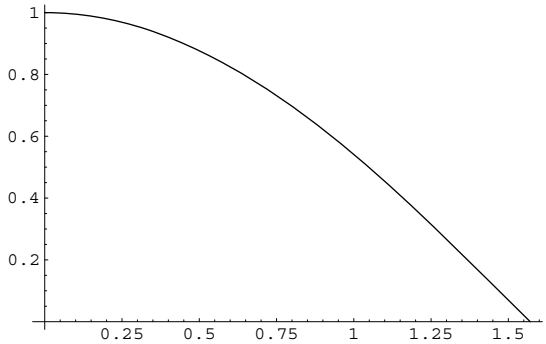
    (a) Do the calculations by hand up through the point where you find the equations to solve simultaneously.

    (b) Use *Mathematica* to do the algebra. **There is a *Mathematica* notebook for you to do this in the usual folder.** You should at least look at the exact answer and appreciate the fact that *Mathematica* can do this. Then have *Mathematica* give numerical approximations for the coordinates. Plot the point on the picture. Give its coordinates and its distance to $(1, 3)$ to three decimal places.

    (c) *Mathematica* gives two answers to this problem. Plot the other point along with its coordinates. What is the significance of this second point?



Problem 11



Problem 12

12. The graph of $y = \cos x$ pictured from 0 to $\pi/2$. Find the length of this curve. Then find the point on the curve halfway between the endpoints as measured along the curve. Give its coordinates to two decimal places. Add the point to the graph.

Selected answers and hints.

1. You can check (a), (d), and (f) with *Mathematica*.

   (c) 0. Be careful not to use L'Hôpital's Rule more than times than justified.

   (e) The answer certainly can't be negative.

2. For a trapezoid sum, you need at least 409 subintervals. $T_{409} = 10.0699$.

3. $8\pi/3$

4. See what *Mathematica* gets for the integral in Cartesian coordinates.

5. $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \frac{50521}{3628800}x^{10} + \ldots$. The derivatives were evalutated using *Mathematica*.

6. $b = 160,000$ works.

7. 1

8. See what *Mathematica* gets.

9. θ should decrease by about 2.73°.

10. This converges conditionally. Remember that for conditional convergence there are two things to prove.

12. The length of the curve is about 1.9101 and the halfway point is about $(.868, .646)$.