Math 112 **Final Exam** **Name:**

13 December 2011 200 Points

"Show enough work to justify your answers."

You may use *Mathematica* on any problem to help you think, however, you may **not** use it as as part of a solution except as noted. If you have trouble with *Mathematica*, please ask.

1. Do all six parts of this problem. (60 points)

   (a) Evaluate: $\int xe^{-2x}\,dx$

   (b) Determine the value of $7 - 5 + 3 - 2 + \frac{4}{3} - \frac{8}{9} + \frac{16}{27} - \frac{32}{81} + \ldots$
   (Note: The pattern doesn't start right at the beginning.)

   (c) Evaluate the following limits.
   $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$ $\lim_{x\to\infty} \frac{\cos x - 1}{x^2}$

   (d) Give the Maclaurin series (the power series centered at 0) for $f(x) = \sin(x^2)$.

   (e) Let $f(x, y) = \cos(x^2 y + x)$. Compute f_x, f_y, and f_{xy}.

   (f) Evaluate: $\int_0^{2/3} \int_0^{\pi x} \cos y\, dy\, dx$

READ THIS !! Do any **seven** of the remaining problems. If you work on more than seven, you will get credit for the best seven. They are worth 20 points each.

$$|I - M_n| \le \frac{K_2(b-a)^3}{24n^2}$$

2. Consider $\int_0^6 \frac{2^x}{3 + e^x}\,dx$. You will need *Mathematica* for this problem.

   (a) Determine, with full reasoning, the number of subintervals to use for a midpoint sum approximation for this integral so that the error will be less than 1/1000. Write enough so that your reasoning is clear. If you use a graph as part of your reasoning, include it. (15 points)

   (b) What is the approximation of the integral based on your answer to part (a)? (Give all decimal places that *Mathematica* provides. (5 points)

3. Evaluate: $\int \frac{1}{\left(4 - x^2\right)^{3/2}}\,dx$

4. Evaluate: $\lim_{x\to\infty} \left(1 - \frac{1}{x}\right)^x$

5. Determine if the following series converges absolutely, converges conditionally, or diverges. Justify your answer.

$$\sum_{k=0}^{\infty} \frac{3^k}{5^k - 8k}$$

6. Give an upper bound on the error made by the approximation indicated below. Express your answer as a fraction that you get by hand and then use *Mathematica* or a calculator to convert it to a decimal.

$$\sum_{k=0}^{\infty} \frac{2^k}{5^k + k} \approx \sum_{k=0}^{8} \frac{2^k}{5^k + k}$$

7. Find the interval of convergence of the following power series, including the endpoints.

$$\sum_{k=1}^{\infty} \frac{(x-5)^k}{k\,3^k}$$

8. Suppose the function f satisfies $f'(x) = (x-2)f(x)$ and $f(0) = 1$.

   (a) Find the first four terms of the Maclaurin series for f (through the x^3 term). We haven't done a problem like this, but it's not difficult. Here is how to proceed. You are given $f(0)$. Use it and the first formula to find $f'(0)$. Differentiate the first formula; use the result and the previous information to find $f''(0)$. Continue. (15 points)

   (b) Use the result of part (a) to approximate $f(.5)$. (5 points)

9. Find the maximum and minimum values of $f(x, y) = xy$ and their locations on the ellipse $4x^2 + y^2 = 8$.

10. Let $f(x) = 4(e^{-x/8} + e^{x/8})$. Write a ready-to-evaluate integral (ask what I mean if you aren't sure) that gives the length of the graph of f for $-2 \le x \le 2$. Use *Mathematica* to evaluate it. Give both *Mathematica*'s representation of the exact value and a numerical approximation. (This is the type of function that models a sagging rope or chain hung from its ends. This integral gives the length of the rope.)

11. Let R be the crescent-shaped region bounded by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ (picture on handout). Express the following integral as a double integral. Your answer should be simplified and ready-to-evaluate. Use *Mathematica* to evaluate it.

$$\iint_R \sqrt{x^2 + y^2}\, dA$$

12. Consider the graph of $y = \sin\sqrt{x}$ for $0 \leq x \leq 5$. If this is rotated around the x-axis, a tall, slender bowl is formed, similar to the bowl of a wine glass (pictures on handout).

   (a) Write a ready-to-evaluate integral that gives the volume of liquid the bowl will hold when filled to the top. Use *Mathematica* to give a numerical approximation.

   (b) Find the depth of the liquid when the bowl is filled to half its total volume. Use *Mathematica* to approximate the depth to two decimal places. Write down the integrals you use *Mathematica* to evaluate that are an essential part of your reasoning (you do not need to include *all* of the guess-and-check integrals).

Have a good break!

Selected answers and hints.

1. (a) See what *Mathematica* gets.
   (b) 19/5
   (c) $-1/2$, 0
   (d) $x^2 - x^6/3! + x^{10}/5! - x^{14}/7! + \ldots$
   (e) $f_{xy} = -2x\sin(x^2y + x) - x^2(2xy + 1)\cos(x^2y + x)$
   (f) $3/(2\pi)$

2. (a) With $K_2 = .09$, get $n > 28.5$, so $n = 29$.
   (b) $M_{29} = 1.78771$ $M_{30} = 1.78769$

3. See what *Mathematica* gets.

4. $1/e$

5. Try this with three different tests: regular comparison test, limit comparison test, and ratio test.

6. .0004369

7. $2 \leq x < 8$

8. (a) $f(x) \approx 1 - 2x + \frac{5}{2}x^2 - \frac{7}{3}x^3$
   (b) $f(.5) \approx .333\cdots = 1/3$

9. The maximum value is 2, which occurs at $(1, 2)$ and $(-1, -2)$.

10. $16\sinh(1/4)$

11. $\frac{1}{3}\left(-4 + 4\pi - 2\sqrt{2}\tanh^{-1}(1/\sqrt{2})\right)$

12. (a) 12.2378
    (b) 2.68