Math 111 **Final Exam** **Name:**

11 December 2007 200 Points

No *Mathematica.*

"Show enough work to justify your answers."

READ CAREFULLY! You must do all parts of Problem 1. After Problem 1, you have a choice of problems.

1. Do all parts of this problem. Each part is worth 10 points. Total: 50 points.

   (a) Use the *definition of derivative* to compute $f'(-2)$ where $f(x) = 3x^2 - 5x + 2$.

   (b) Find the maximum and minimum values of $f(x) = x^3 - 27x + 5$ and their locations on the interval $[-4, 1]$. Be clear in your answer which are values of f and which are their locations.

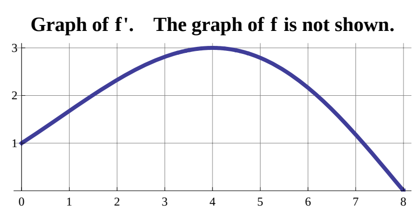
   (c) Consider the curve $x^2 - 2xy + 5y^2 = 80$. Verify that the point $(-6, 2)$ is on the curve and find an equation of the line tangent to the curve at this point.

   (d) Evaluate the following, giving the exact, simplified value. $\displaystyle\int_{-\pi/12}^{3\pi/8} \cos 2x \, dx$

   (e) Evaluate the following limit. You may not use L'Hôpital's Rule or high school "tricks." $\displaystyle\lim_{x\to\infty} \frac{3x^3 - 500x^2 + 5x - 7}{2 - 25x + 200x^2 - 8x^3}$

READ CAREFULLY! Do any **ten** of the remaining problems. If you work on more than ten, you will get credit for the best ten.

2. The graph of the derivative of a function f is shown. Suppose that $f(2) = 5$. Use the Speed Limit Law to find numbers L and U such that $L \le f(6) \le U$. For full credit, L and U must not differ by more than 5. (For extra credit, find values of L and U that differ by less than 3. Hint for extra credit: Think integration.)



3. Sketch a convincing graph of a function with all of the following properties. (You might make a rough sketch on scratch paper and then copy it to the exam.)

f is an odd function $\quad$ f is continuous everywhere $\quad$ f is differentiable everywhere except $x = 0$

$\displaystyle\lim_{x\to 0} f'(x) = \infty \qquad f'(2) = 0 \qquad f''(x) > 0$ for $x > 3 \qquad \displaystyle\lim_{x\to\infty} f(x) = 1$

4. You are given a disk of radius 1. You are to cut a circular wedge from the disk and attach the cut edges together to form a cone. Find the height, radius, and volume of the cone of maximal volume.

5. Find, with careful explanations, the maximum and minimum values of the function $f(x) = x^4 - 4x^3$. (It is possible that one or both might not exist. This also requires a careful explanation.)

6. The general solution of $y'' = ky$ for $k > 0$ is $y(t) = Ae^{\sqrt{k}t} + Be^{-\sqrt{k}t}$. Find the particular solution of $y'' = 9y$ for which $y(0) = -5$ and $y'(0) = 3$.

7. The graph of a function f defined on $[0, 8]$ is pictured. The graph is symmetric about $t = 4$ and the regions labeled A and B have the same area. Let $F(x) = \int_0^x f(t)\,dt$ for $0 \le x \le 8$. Answer questions about the graph of F. No explanations are needed. **Note: The graph of F is not shown.**

   (a) On which subintervals is $F(x) \ge 0$?

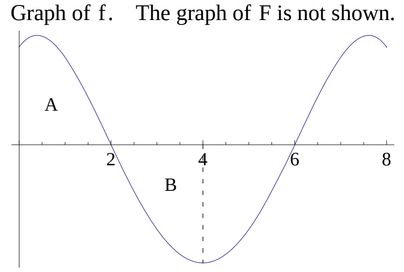
   (b) On which subintervals is $F(x)$ increasing?

   (c) On which subintervals is $F(x)$ concave up?

   (d) Where does F have its maximum?

   (e) Where does F have local maxima?

Graph of f. The graph of F is not shown.



8. Suppose f is a continuous function on the interval $[a, b]$. The expressions $\int_a^b f(x)\,dx$ and $\int f(x)\,dx\big|_a^b$ look similar, but they have very different meanings.

   (a) Describe the meaning of each without using the words "integral" or "integrate." (Think about the context in which we first encountered $\int_a^b f(x)\,dx$. It may help to think about $\int f(x)\,dx\big|_a^b$ as a two-step process, namely $F(x) = \int f(x)\,dx$ and $F(x)\big|_a^b$.)

   (b) What does the Fundamental Theorem of Calculus say about how their values are related?

9. Consider the integral $\displaystyle\int_0^2 \sqrt{4 - x^2}\,dx$.

   (a) Compute L_4, R_4, and T_4, that is, the left endpoint, right endpoint, and trapezoid approximations for the integral with four subintervals. Express your results to four decimal places. Show enough work so I can tell what you are thinking.

   (b) Is T_4 an overestimate or underestimate of the true value of the integral? How can you tell?

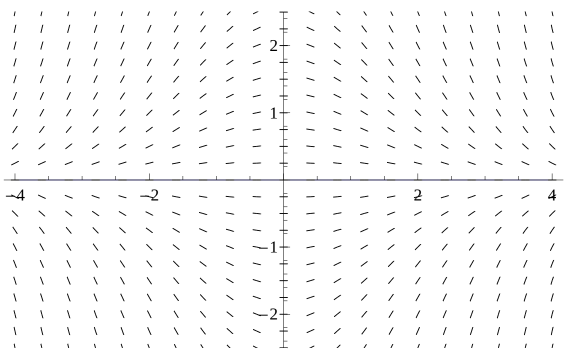
10. Evaluate the following using a substitution. $\displaystyle\int \sin x\sqrt{\cos x + 2}\,dx$

11. The pictured slope field is for the ODE $y' = -ty/2$.

   (a) Draw the solution curve for the solution with $y(0) = 1$. Label it (a).

   (b) Draw the solution curve for the solution with $y(2) = -1$. Label it (b).

   (c) If $y(t)$ is a solution, what is $\lim_{t\to\infty} y(t)$?



   (d) Determine if $y(t) = e^{-t^2}$ is a solution of the ODE.

12. Let $f(x) = e^{-x}$.

   (a) Determine the quadratic approximation $q(x)$ (or $p_2(x)$) for $f(x) = e^{-x}$ for $x_0 = 0$.

   (b) Use your answer to part (a) to approximate $e^{.25}$. Give your answer to five decimal places.

13. Consider the parametric curve defined by $x = 3 + 2\sin t$, $y = 2 + 2\cos t$, for $0 \leq t \leq 3\pi/2$.

   (a) Draw the curve (note the range of t-values). Label the points of the curve corresponding to $t = 0$, $\pi/2$, π, $3\pi/2$ with their t-values.

   (b) Eliminate the parameter, that is, find one equation in x and y without t. Hint: Solve for $\sin t$ and $\cos t$ and use a trig identity.

14. A bicyclist is riding north at 10 miles per hour on County Road A towards the intersection of County Road A and County Road B. A second bicyclist is riding east at 10 miles per hour on County Road B away from the intersection. At the same moment the first bicyclist is 3 miles from the intersection and the second bicyclist is 2 miles from the intersection. How fast is the distance between them changing, and is it increasing or decreasing?

15. Suppose that f is a function defined on $[0, 2]$ that has a maximum value, but no minimum value. What does the Extreme Value Theorem imply about f? (You might write down the complete statement of the EVT to help you think.) Draw a *clear* graph of such a function.

Selected answers and hints.

1. Some of these are BSE problems. You do remember the BSE, don't you?

   (b) The minimum value is -21 and its location is $x = 1$. The most common error on this was not knowing the difference between value and location. Values of a function refer to its outputs. Locations refer to its inputs. Note that $(1, -21)$ is neither a value nor a location; it is the lowest point on the graph.

   (c) To reduce the amount of algebra, plug in the point before solving for y'. The slope is $1/2$.

   (d) You need to be able to use the unit circle to figure out the values of $\sin x$ and $\cos x$ for the standard angles! $(\sqrt{2}+1)/4$.

2. $13 \le f(6) \le 17$. The hint for the extra credit leads to $10 \le f(6) - f(2) \le 12$.

4. $h = 1/\sqrt{3}$, $r = \sqrt{2/3}$, $V = 2\sqrt{3}\pi/27$. Some points were taken off for giving approximations instead of the exact answers.

5. Be sure your procedure for finding maxima and minima is correct! One common error was to conclude that the maximum value was 0; it's not. Another common error was to miss the critical point at $x = 0$. This is caused by dividing by x instead of factoring it out. Never divide by an unknown unless you are sure it is not zero. Another common error was using the basic second derivative test. The basic second derivative test tells you about local extrema, but not global extrema.

6. The most common error was showing that $y(t) = Ae^{\sqrt{k}t} + Be^{-\sqrt{k}t}$ is a solution of $y'' = ky$. The problem doesn't ask for that. $y(t) = -2e^{3t} - 3e^{-3t}$

7. The most common error was to answer the questions for the graph shown, even though it is not the graph of F. (a) $[0, 4]$, (c) $[0, .5]$ and $[4, 7.5]$, approximately, (e) at $x = 2$ and $x = 8$.

8. We started talking about $\int_a^b f(x)\,dx$ in Chapter 5. We started talking about $\int f(x)\,dx$ in Chapter 3, although we didn't use this notation until Chapter 5.

9. (a) $L_4 = 3.4955$. To save time, remember how T_4 is related to L_4 and R_4. (b) This is related to the concavity of the function.

10. See what *Mathematica* gets.

11. (d) It is not a solution.

12. (a) $q(x) = 1 - x + x^2/2$ (b) $e^{.25} \approx q(-.25) = 1.28125$ (compare with the actual value of $e^{.25}$ to five places: 1.28402)

13. The answer to (b) is the equation of the circle of radius 2 centered at $(3, 2)$. The curve for (a) is $3/4$ of this circle.

14. The bicyclists are getting closer at 2.77 mph.

15. f must be discontinuous somewhere.