

13 December 2006

200 Points

No *Mathematica*. No calculators except as indicated.*"Show enough work to justify your answers."*

Note: Exact answers are expected except in problems involving approximations.

Part I. Do all parts of problems 1 through 4. Do your work in the space provided. Total 80 points.

1. State the definition of $f'(a)$ and use it to compute $f'(3)$ for $f(x) = x^2 - 2x$. (10 points)
2. Compute derivatives of the following functions and simplify. (10 points each)

(a) $f(x) = \cos(x^2) + \sqrt[3]{5x+7} + \sin 2$

(b) $g(y) = \frac{y^2 - 3}{y^3 + 5} + y \arctan(2y)$

3. Evaluate the following limits. Justify your answers. (10 points each)

(a) $\lim_{x \rightarrow \infty} \frac{3x^2 - 5x + 7}{3 - 2x^2}$

(b) $\lim_{x \rightarrow 2} \frac{x}{x^2 - 4}$

4. Evaluate the following integrals and antiderivatives. (10 points each)

(a) $\int \left(\sin(2x) + \frac{1}{e^{3x}} \right) dx$

(b) $\int \left(\frac{3}{x} - \frac{2}{1+x^2} \right) dx$

(c) $\int_1^3 \left(x^2 + \frac{1}{x^2} \right) dx$

Part II. Do any **eight** of the remaining problems. If you do more than eight, you will get credit for the best eight. **Do each problem on a separate sheet of paper.** Put the problem number in the upper right corner of the page. Avoid writing in the upper left corner where the staple will go. (15 points each). Total 120.

5. Evaluate: $\int_0^3 x\sqrt{9-x^2} dx$

6. Find two values of θ between 0 and 2π for which $\sin \theta = 0.6$. Express your answer to four decimal places. You will need to use a calculator. Part of your explanation should use the unit circle.

7. The following table gives some values of a function f . (You may use a calculator on this problem. Give enough detail so I can tell what you are thinking.)

x	0.7	0.8	0.9	1.0	1.1	1.2	1.3
$f(x)$	0.683	0.717	0.783	0.841	0.891	0.932	1.035

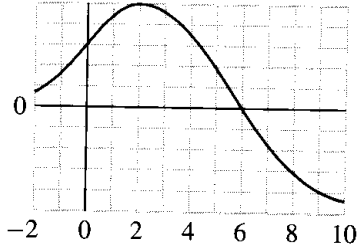
- (a) Based on this, give a reasonable approximation of $f'(1)$. (5 points)
- (b) Give a right sum approximation of $\int_{.8}^{1.2} f(x) dx$ using four subintervals. (7 points)
- (c) Does the approximation in (b) overestimate or underestimate the true value of the integral? What feature of the function causes this (increasing, decreasing, concave up, concave down)? (3 points)
8. A box is being constructed to have a volume of 72 cubic feet. The length of the box is to be two times its width. Find the dimensions of the **two** boxes, one that **minimizes** the surface area, and one that **maximizes** the surface area. Be sure to include an argument that you have found the minimum and maximum.
9. A bug is crawling up a wall at 2 inches per second. A cat is stalking the bug and is moving towards the wall at 6 inches per second. At a particular moment the bug is 4 feet above the floor and the cat is 3 feet from the wall. Determine how fast the distance between them is changing and if the cat is getting closer to the bug or farther from it at that moment. State your answer in a sentence.
10. A function f and its first two derivatives are defined and continuous for all values of x and they have the following properties:

$$f(0) = 4 \quad \text{and} \quad f(2) = 0, \quad f'(0) = f'(2) = 0$$
$$f''(x) > 0 \text{ for } x > 1 \quad \text{and} \quad f''(x) < 0 \text{ for } x < 1$$

Based on this information, sketch a meaningful graph of f . Indicate (label) all critical points, local maxima and minima, inflection points, and asymptotes.

11. Find the value of b so that the area in the first quadrant under the graph of $y = x^2$ between $x = 1$ and $x = b$ is 5.
12. Let $f(x) = \sqrt{x}$. Use an appropriate quadratic approximation $q(x)$ of f to approximate $\sqrt{8.3}$. You should write a simplified (*not* multiplied out!) formula for $q(x)$, and you may use a calculator to do the arithmetic.

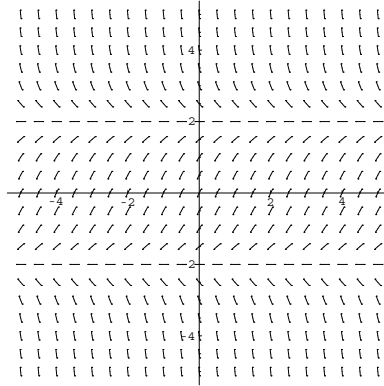
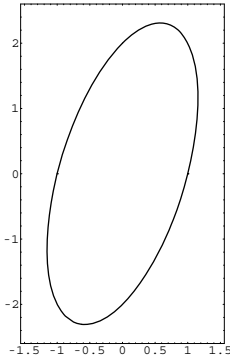
13. The graph of the derivative of f is shown. Use the Speed Limit Law to find numbers L and U such that $L \leq f(4) - f(0) \leq U$. Carefully explain. For full credit, L and U must differ by less than 10. You may earn five points extra credit by finding values for L and U that differ by less than 8, with a careful explanation.



Graph of f' . **The graph of f is not shown.**

14. Suppose f is a function defined for all real numbers. Suppose that $f'(x) < 0$ and $f''(x) > 0$ for all x . Let $L(x)$ be the function whose graph is the tangent line for f at 2. Rank the following in increasing order: $f(0)$, $f(2)$, $f(3)$, $L(0)$, $L(2)$, $L(3)$. Of course, you should explain. It helps to draw some possible graphs.
15. Newton's Law of Cooling says that the rate at which the temperature of an object changes is proportional to the difference between the object's temperature and the temperature of the surrounding air.
- Write a differential equation that expresses this law. Use T for the temperature of the object (a cup of coffee or a cold soda), and assume that the temperature of the air is a constant 70° .
 - Show that $T(t) = 70 + Ae^{-kt}$ is a solution of the differential equation. Here A is an arbitrary constant and k is a constant that should appear in your differential equation. Note: If you have written your differential equation in another form, your solution may be $T(t) = 70 + Ae^{kt}$.
 - If $T(0) = 170$ and $T'(0) = -1/5$, what are the values of A and k ?

16. The ellipse $4x^2 - 2xy + y^2 = 4$ is pictured below.
- Determine the coordinates of the point on the ellipse where it crosses the positive x -axis. (3 points)
 - Find an equation of the line that is tangent to the curve at this point. For full credit use the point-slope formula. (9 points)
 - Add the tangent line to the picture. (3 points)



17. The slope field pictured above is for the differential equation $y' = (2 + y)(2 - y)$.
- If $y(t)$ is the solution of the differential equation such that $y(-2) = 0$, what is $y'(-2)$?
 - Draw the graph of this solution on the slope field.
 - For this solution, what is $\lim_{t \rightarrow \infty} y(t)$?

18. Suppose that f is a function such that

$$f(x)^2 = \int_0^x f'(t)^2 dt + 2006$$

for all x . Prove that $f'(x) = 2f(x)$.

Happy Holidays!

Selected answers and hints.

6. .6435 and 2.4981 radians.
7. (a) .540 (b) .3447
8. The minimum surface area is 108 square inches, but this doesn't answer the question!
9. The distance between the cat and the bug is changing at 2 inches per second. (This is not the complete answer.)
11. $b = \sqrt[3]{16}$
12. $f(8.3) \approx q(8.3) = 2.88106$
13. $12 \leq f(4) - f(0) \leq 20$. A better estimate is $14.5 \leq f(4) - f(0) \leq 20$, which you can get by doing two estimates, one for $0 \leq x \leq 2$ and one for $2 \leq x \leq 4$, and combining them. An even better estimate is $16 \leq f(4) - f(0) \leq 19$, which you can get by thinking of $f(4) - f(0)$ as the area under the graph!
15. $A = 100$. k is $1/500$ or $-1/500$, depending on how you express the differential equation.
16. (a) It's okay to guess, but if you do, you must verify your guess.
(b) $y = 4(x - 1)$
17. (a) 4 (c) 2
18. Differentiate both sides.