Math 111 **Final Exam** **Foote**

18 December 2002 200 Points

"Show enough work to justify your answers."

Read Carefully! This exam has two parts. In Part I you are to work all of the problems, and you should write your answers on the exam. In Part II you are to work at least six of the problems, and you are to work each problem on a separate sheet of blank paper.

Part I. Do **all eleven** of the problems in this part. Show all steps of every computation. Your answers must be independent of *Mathematica*. (10 points each)

1. Let $f(x) = 1 - 2x^2 - 3x$. Use the definition of the derivative to show that $f'(-2) = 5$.

2. Let $f(x) = 4x^3 - 15x^2$. Find the maximum and minimum values of f and their locations on the interval $[1, 4]$. In your answer you should clearly indicate which are the values and which are the locations.

3. Evaluate $\displaystyle\int_{-1}^{3} (x^2 - x + 1)\, dx$.

4. Evaluate $\displaystyle\lim_{x \to 3} \frac{x - 3}{x^2 - x - 6}$. You may not use L'Hopital's Rule.

5. Evaluate $\displaystyle\lim_{x \to \infty} \frac{x^2 - 3}{3x^2 - \sqrt{x} - 6}$. You may not use L'Hopital's Rule.

6. Verify that $\displaystyle\int \arctan x\, dx = x \arctan x - \frac{1}{2} \ln(1 + x^2) + C$.

7. Suppose that $f'(x) = e^{2x} + \sin x + \sqrt{3}$ and that $f(0) = 7/2$. Find a formula for $f(x)$.

8. Evaluate $\displaystyle\int \frac{x}{\sqrt{2x + 1}}\, dx$ using the substitution $u = 2x + 1$.

9. Let $g(t) = t^2 \sec(t^2) + \sqrt[3]{t^4} + \ln 19$. Compute $g'(t)$.

10. Determine if $y = x^2 - \dfrac{1}{x}$ is a solution of the differential equation $xy' = x^2 + y$.

11. Consider the curve defined by $x^2 + xy + y^2 = 1$. Verify that the point $(0, 1)$ is on this curve and then find the equation line tangent to the curve at that point.

Part II. Do any **six** of the problems in this part. If you do more than six, you will get credit for the best six. Note that some problems have multiple parts. You may use *Mathematica* in your solutions, except as indicated. If you use *Mathematica*, be very clear what you use it for, and show enough work that I can tell what you are thinking. If you have any question about what constitutes sufficient work, be sure to ask. If you have trouble with *Mathematica*, be sure to ask for help. (15 points each)

Please do each problem on a separate sheet of paper. Put the problem number in the upper **right** corner of the page. Avoid writing anything in the upper left corner (to leave room for a staple).

12. Suppose that f is a function, that $f(3) = 5$, and that $2 \quad f'(x) \quad 3$ for all x. Give upper and lower estimates for $f(7)$. Explain. For full credit, your upper and lower estimates must be as close to each other as the information allows.

13. A box without a lid is to be constructed from an 8in by 8in square piece of cardboard by cutting small squares out of the corners and folding up the sides. What are the largest and smallest possible volumes for such a box, and how big should the cutouts be to get these volumes?

14. The following table gives values of a function f for some values of its input.

x	0	0.5	1.	1.5	2
$f(x)$	0	1.8	3.1	3.6	4.2

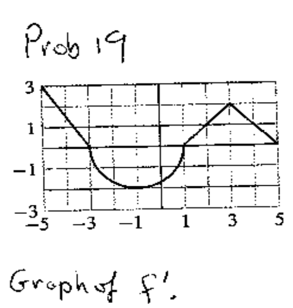
   a) Use the data to give an approximation for $f'(1.25)$. Show enough so I can tell what you are doing.
   b) If you are sitting in a left seat, use L_3 to approximate $\int_{1/2}^{2} f(x)\,dx$.
   If you are sitting in a right seat, use R_3 to approximate $\int_{0}^{3/2} f(x)\,dx$.
   Note that these integrals are different! Show enough so I can tell what you are doing.
   c) Do you think that your approximation overestimates or underestimates the true value of the integral? What feature of the function causes this?

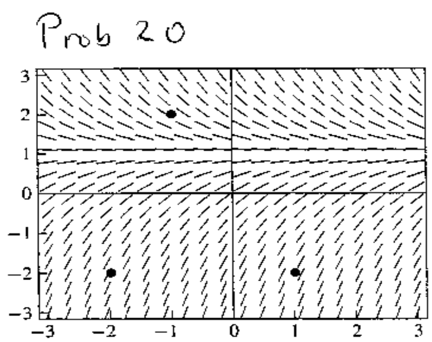
15. Draw the graph of a function f with the following properties. Be sure to draw the scale on your coordinate axes so that all relevant parts of the graph are easily visible.

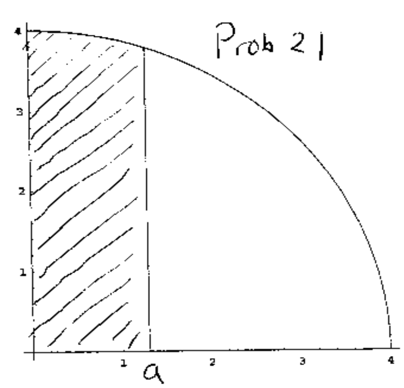
$$f \text{ has a vertical asymptote at } x = 0, \qquad f(2) = -3,$$
$$f'(x) < 0 \text{ for } 0 < x < 3, \qquad f'(x) > 0 \text{ for } x > 3, \qquad f''(x) < 0 \text{ for } x > 5.$$

16. Newton's Law of Cooling says that the rate at which the temperature of an object changes is proportional to the difference between the object's temperature and the temperature of the surrounding air.
   a) Write a differential equation that expresses this law. Use T for the temperature of the object (a cup of coffee or a cold soda), and assume that the temperature of the air is a constant 70°.
   b) Show that $T(t) = 70 + Ae^{-kt}$ is a solution of the differential equation. Here A is an arbitrary constant and k is a constant that should appear in your differential equation. Note: If you have written your differential equation in another form, your solution may be $T = 70 + Ae^{kt}$.
   c) What is $\lim_{t\to\infty} T(t)$?

17. Use a suitable quadratic approximation of $f(x) = \sqrt{x}$ to approximate $\sqrt{34}$.
18. Evaluate $\int \tan x \, dx$. Hint: Express $\tan x$ in terms of $\sin x$ and $\cos x$.
19. The graph of the derivative of a function f is shown.
    Answer the following. **Note: The graph of f is not shown.** (5 points each)
    a) Indicate the locations of the local maxima and local minima of f, indicating which is which. Don't forget the endpoints. Briefly indicate how you know.
    b) If $f(3) = 5$, what is the exact value of $f(-1)$? Note: Since we have done integration, we have a precise way of answering this.
    c) Does the graph of f lie above or below its tangent line at $x = 1$? How do you know?
20. The pictured slope field is for one of the differential equations $y' = y(1-y)$ or $y' = 1-y$.
    a) Which differential equation corresponds to the slope field. Briefly explain how you know.
    b) If $y(t)$ is the solution of the differential equation such that $y(-1) = 3$, what is $y'(-1)$?
    c) On the picture below, draw the graph of this solution on the slope field.
    d) If $y(t)$ is any solution of the differential equation, what is $\lim_{t\to\infty} y(t)$?
21. The pictured curve is the portion of the circle $x^2 + y^2 = 16$ in the first quadrant. The value of a determines the the amount of area that is shaded area. Note: This problem makes a fair amount of use of *Mathematica*. If you need help with *Mathematica*, be sure to ask. You are responsible for ensuring that the answers *Mathematica* gets are reasonable answers for the questions (if it gets wrong answers because you mistype something, you need to catch it).
    a) Write an integral that expresses the shaded area in terms of the variable a. Use *Mathematica* to evaluate the integral, and write the result on your paper.
    b) Check that the expression you get in part a) gives the correct area for the smallest and largest values of a that make sense in the problem.
    c) Find an approximation for the value of a, good to at least three decimal places so that half of the area of the quarter circle is shaded. You may do this by guess and check, or for **ten points extra credit**, by using Newton's Method. In either case, you **must** write enough on your exam so that I can tell what you did with *Mathematica* and how you drew your conclusions.







Selected answers and hints.

2. The maximum value is 16.
8. After substituting, break the fraction into the sum of two fractions.
11. Remember that the slope of a line is constant!
12. The lower bound is 13.
13. The maximum volume is approximately 37.926 cubic inches.
14. a) $f'(1.25) \approx 1$
    b,c) $\int_{1/2}^{2} f(x)\,dx \approx L_3 = 4.25$, which is an underestimate. Why? (It doesn't involve concavity.)
16. a) dT/dt is proportional to $T - 70$ and so $dT/dt = k(T - 70)$ or $dT/dt = -k(T - 70)$.
    c) This is asking what the temperature approaches as time passes.
17. $\sqrt{x} \approx q(x) = 6 + \frac{1}{12}(x - 36) - \frac{1}{1728}(x - 36)^2$, $\sqrt{34} \approx q(34) \approx 5.831$
18. $\ln|\sec x| + C$
19. a) There are two local maxima and two local minima.
    b) $\pi + 3$
    c) The graph of f lies above its tangent line near $x = 1$. Why?
20. a) $y' = 1 - y$ b) $y'(-1) = -2$ d) 1
21. a) The integral for the area as a function of a evaluates to $A(a) = \frac{a}{2}\sqrt{16 - a^2} + 8\arcsin(a/4)$.
    b) Check that $A(0) = 0$ and $A(4) = 4\pi$.
    c) $a \approx 1.61589$