CONVERGENCE TESTS FOR SERIES: COMMENTS AND PROOFS
PART IV: THE ALTERNATING SERIES TEST

MATH 112

The convergence tests for series have nice intuitive reasons why they work, and these are fairly easy to turn into rigorous proofs. In these notes we will prove the standard convergence tests and give two tests that aren't in our text.

It's important to remember that the convergence or divergence of a series depends only on what happens to some tail of the series—the inclusion or omission of a finite number of terms cannot change a convergent series into a divergent one or vice versa. (If the series converges, the sum is affected, of course.) Because of this, we can let a little convenient sloppiness into our notation. When it doesn't matter what the starting point is for a series, we can write $\sum a_k$ instead of $\sum_{k=1}^{\infty} a_k$ (or $\sum_{k=5}^{\infty} a_k$ or $\sum_{k=k_0}^{\infty} a_k$). More specifically, the notation $\sum a_k$ will always mean an infinite sum, and will only be used when the starting point for the sum is not important.

THE ALTERNATING SERIES TEST

Theorem (Alternating Series Test). *Let $\sum a_k$ be a series. Suppose there is an index K such that the tail $\sum_{k=K}^{\infty} a_k$ is alternating, and*

(1) *The sequence $|a_K|$, $|a_{K+1}|$, $|a_{K+2}|$, $\ldots$ is decreasing, and*

(2) $\lim\limits_{k\to\infty} a_k = 0$.

Then the series converges.

Intuitive reason why this is true. Consider the partial sums of $\sum_{k=K}^{\infty} a_k$. Each time a positive term is added the sum moves to the right; each time a negative term is added the sum moves to the left. Because of (1), each time the sum moves, it moves less than it did the previous time. Because of (2), the amount that it moves goes to zero. The partial sums are moving back and forth, getting closer and closer to each other. It just *has* to converge!

Proof. For convenience, let $b_k = |a_k|$ for $k \geq K$. Suppose that $a_k > 0$ when k is even and $a_k < 0$ when k is odd. Then $\sum_{k=K}^{\infty} a_k = \sum_{k=K}^{\infty} (-1)^k b_k$. (If $a_k < 0$ when k is even and $a_k > 0$ when k is odd, we get an extra factor of -1, and the rest of the proof has to be modified slightly.)

Let $s_n = \sum_{k=K}^{n} (-1)^k b_k$. It will be convenient to keep track of the partial sums with even and odd indices separately. Let J be the smallest integer such that $2J \geq K$. Then let $c_j = s_{2j}$ and $d_j = s_{2j+1}$ for $j \geq J$. Now we make two observations. First,

$$d_j = d_{j-1} + b_{2j} - b_{2j+1} \geq d_{j-1} \text{ for all } j \geq J, \text{ since } b_{2j+1} \geq b_{2j} \text{ by (1).}$$

Thus d_j, $j = J, J+1, \ldots,$ is an increasing sequence. Similarly c_j, $j = J, J+1, \ldots,$ is decreasing. Second,

$$d_j = c_j - b_{2j+1} < c_j \text{ for all } j \geq J, \text{ since } b_{2j+1} > 0.$$

Since $c_j \leq c_J$, we have that $d_j < c_J$ for all $j \geq J$, that is the sequence d_j, $j = J, J+1, \ldots,$ is bounded above. Similarly c_j, $j = J, J+1, \ldots,$ is bounded below. By Lemma 1 in Part I of these notes (it really is a useful lemma!) both sequences converge. Let $C = \lim_{j\to\infty} c_j = \lim_{j\to\infty} s_{2j}$ and $D = \lim_{j\to\infty} d_j = \lim_{j\to\infty} s_{2j+1}$. Could these limits be different? We have $D - C = \lim_{j\to\infty} s_{2j+1} - s_{2j} = \lim_{j\to\infty} b_{2j+1} = 0$ by (2), and so $D = C$. Since the two subsequences of partial sums both have the same limit, their common value is the limit of the full sequence of partial sums, $\lim_{n\to\infty} s_n = C = D$, and so the series converges.

Extra Credit Challenge Problems.

(1) Let c_j and d_j be as in the proof of the alternating series test. Prove that every term of the c-sequence is greater than every term of the d-sequence, that is, $c_j > d_i$ for all $i, j \geq J$.

(2) Find an example of a divergent alternating series $\sum a_k$ for which $\lim_{k\to\infty} a_k = 0$.

(3) Find an example of a convergent alternating series $\sum a_k$ for which no tail of the sequence $|a_1|$, $|a_2|$, $|a_3|$, ... is decreasing.

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