CONVERGENCE TESTS FOR SERIES: COMMENTS AND PROOFS PART III: THE LIMIT COMPARISON AND ROOT TESTS

MATH 112

The convergence tests for series have nice intuitive reasons why they work, and these are fairly easy to turn into rigorous proofs. In these notes we will prove the standard convergence tests and give two tests that aren't in our text.

It's important to remember that the convergence or divergence of a series depends only on what happens to some tail of the series—the inclusion or omission of a finite number of terms cannot change a convergent series into a divergent one or vice versa. (If the series converges, the sum is affected, of course.) Because of this, we can let a little convenient sloppiness into our notation. When it doesn't matter what the starting point is for a series, we can write $\sum a_k$ instead of $\sum_{k=1}^{\infty} a_k$ (or $\sum_{k=5}^{\infty} a_k$ or $\sum_{k=k_0}^{\infty} a_k$). More specifically, the notation $\sum a_k$ will always mean an infinite sum, and will only be used when the starting point for the sum is not important.

TWO ADDITIONAL CONVERGENCE TESTS: THE LIMIT COMPARISON TEST AND THE ROOT TEST

Theorem (Limit Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series of positive terms. Let $\ell = \lim\limits_{k\to\infty} \dfrac{a_k}{b_k}$, provided this limit exists (∞ is allowed as a possible value). If ℓ is neither 0 nor ∞, then either both series converge or both series diverge.*

The limit comparison test often works in situations when you would want to use the regular comparison test. It is easier to apply since you don't have to prove an inequality. The down side is that you then don't have an inequality with which to make error estimates, which is why the authors of our text left it out.

Intuitive reason why this is true. Since $\lim\limits_{k\to\infty} a_k/b_k = \ell$, we have that $b_k \approx \ell a_k$ for sufficiently large k. The tails of the series then satisfy $\sum b_k \approx \sum \ell a_k = \ell \sum a_k$. Since ℓ is neither 0 nor ∞, we have that $\sum a_k$ and $\sum b_k$ are either both finite or infinite.

Proof of half of the Limit Comparison Test. Similar to the proof of the ratio test, the idea is to show that $b_k \leq La_k$ where L is some number bigger than ℓ, and then to use the regular comparison test. Let L be any number larger than ℓ. Since $\lim_{k\to\infty} a_k/b_k = \ell$, there exists an index K such that $a_k/b_k < L$ for all $k \geq K$, that is, $a_k < Lb_k$. (Note that we have used the assumption that ℓ is not ∞ here.)

Now, if $\sum b_k$ converges, so does $\sum Lb_k$, and then $\sum a_k$ converges by the regular comparison test.

Theorem (Root Test). *Let $\sum a_k$ be a series of positive terms. Let $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$, provided this limit exists. There are three possibilities.*

(1) *If $0 \leq \rho < 1$, then the series converges.*
(2) *If $\rho > 1$, then the series diverges.*
(3) *If $\rho = 1$, then the test fails.*

Intuitive reason why this is true. For large k we have $\sqrt[k]{a_k} \approx \rho$ or $a_k \approx \rho^k$. If a_k were exactly equal to ρ^k, the series would be geometric with ratio ρ. Thus some tail of the series is nearly geometric with ratio ρ, and so conclusions (1) and (2) are plausible.

There is more than a passing similarity between the ratio and root tests. The proofs are very similar, as can be guessed by the intuitive reasoning. Furthermore, for a given series, the two values of ρ are the same, that is $\lim_{k\to\infty} a_{k+1}/a_k = \lim_{k\to\infty} \sqrt[k]{a_k}$, provided both limits exist. The root test is sometimes easier to use in situations in which you would otherwise use the ratio test.

Extra Credit Challenge Problems.

(1) Prove the other half of the limit comparison test, that is, if $\sum b_k$ converges, so does $\sum b_k$.
(2) Even if ℓ is 0 or ∞ in the limit comparison test, some conclusions can be drawn. Formulate and prove two statements. Here is one: If $\ell = 0$ and (*one of the series goes here*) converges, then so does (*the other series goes here*).
(3) Prove the root test.

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