

CONVERGENCE TESTS FOR SERIES: COMMENTS AND PROOFS

PART II: THE RATIO TEST

MATH 112

The convergence tests for series have nice intuitive reasons why they work, and these are fairly easy to turn into rigorous proofs. In these notes we will prove the standard convergence tests and give two tests that aren't in our text.

It's important to remember that the convergence or divergence of a series depends only on what happens to some tail of the series—the inclusion or omission of a finite number of terms cannot change a convergent series into a divergent one or vice versa. (If the series converges, the sum is affected, of course.) Because of this, we can let a little convenient sloppiness into our notation. When it doesn't matter what the starting point is for a series, we can write $\sum a_k$ instead of $\sum_{k=1}^{\infty} a_k$ (or $\sum_{k=5}^{\infty} a_k$ or $\sum_{k=k_0}^{\infty} a_k$). More specifically, the notation $\sum a_k$ will always mean an infinite sum, and will only be used when the starting point for the sum is not important.

THE RATIO TEST

Theorem (Ratio Test). *Let $\sum a_k$ be a series of positive terms. Let $\rho = \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k}$, provided the limit exists. There are three possibilities.*

- (1) *If $0 \leq \rho < 1$, then the series converges.*
- (2) *If $\rho > 1$, then the series diverges.*
- (3) *If $\rho = 1$, then the test fails.*

Intuitive reason why this is true. For large k we have $a_{k+1}/a_k \approx \rho$ or $a_{k+1} \approx \rho a_k$. If a_{k+1} were exactly equal to ρa_k , the series would be geometric with ratio ρ . Thus some tail of

the series is nearly geometric with ratio ρ , and so conclusions (1) and (2) are plausible.

Proof. Suppose $\rho < 1$. The idea is to compare the series with a geometric series with a ratio slightly larger than ρ . Let r be any number such that $\rho < r < 1$. Since $\lim_{k \rightarrow \infty} a_{k+1}/a_k = \rho$, there is some index K such that $a_{k+1}/a_k < r$ for all $k \geq K$. Let $b = a_K$, and then define $b_K = b$, $b_{K+1} = br$, $b_{K+2} = br^2$, and in general, $b_{K+j} = br^j$, which can be written as $b_k = br^{k-K}$ for $k \geq K$. Then $\sum_{k=K}^{\infty} b_k = \sum_{k=K}^{\infty} br^{k-K} = \sum_{k=0}^{\infty} br^k$ is a convergent geometric series. Note that we have

$$\begin{aligned} a_K &= b = b_K, \\ a_{K+1} &< ra_K = rb = b_{K+1}, \\ a_{K+2} &< ra_{K+1} < r^2b = b_{K+2}, \\ &\vdots \\ a_{K+j} &< ra_{K+j-1} < r^j b = b_{K+j}, \end{aligned}$$

and so $a_k \leq b_k$ for each $k \geq K$. Then $\sum a_k$ converges by the comparison test.

The case when $\rho > 1$ is similar. Let r be a number such that $1 < r < \rho$. Since $\lim_{k \rightarrow \infty} a_{k+1}/a_k = \rho$, there is some index K such that $a_{k+1}/a_k > r$ for all $k \geq K$. We could compare $\sum a_k$ with a divergent geometric series, but there's something easier. For $k \geq K$ we have $a_{k+1} > ra_k > a_k$, since $r > 1$. Thus the terms of the series are increasing. Hence they don't go to zero, and so the series diverges.

To say that the test fails when $\rho = 1$ means that there are examples of series for which $\rho = 1$, some of which converge and some of which diverge. The proof consists of finding one of each, which you should do!