

## PROOF OF THE TRAPEZOID SUM ERROR ESTIMATE

MATH 112

**Theorem.** Suppose that  $f: [a, b] \rightarrow \mathbb{R}$  is twice-differentiable. Let

$$T_n = \sum_{k=1}^n \frac{f(x_{k-1}) + f(x_k)}{2} \Delta x = \left( \frac{f(x_0)}{2} + \sum_{k=1}^{n-1} f(x_k) + \frac{f(x_n)}{2} \right) \Delta x,$$

where  $n$  is a positive integer,  $\Delta x = (b - a)/n$ , and  $x_k = a + k\Delta x$  for  $k = 0, 1, \dots, n$ . Let  $K_2$  be any number such that  $|f''(x)| \leq K_2$  for all  $x$  with  $a \leq x \leq b$ . Then

$$(1) \quad \left| T_n - \int_a^b f(x) dx \right| \leq \frac{K_2(b-a)^3}{12n^2},$$

The proof uses a basic property of integration called *monotonicity*: If  $f, g: [a, b] \rightarrow \mathbb{R}$  are integrable and  $f(x) \leq g(x)$  for all  $x$  in  $[a, b]$ , then  $\int_a^b f(x) dx \leq \int_a^b g(x) dx$ . When  $f$  and  $g$  are positive functions, this just says that the area under the graph of  $f$  is no greater than the area under the graph of  $g$ .

*Proof of the error estimate.* First we will prove (1) for one interval, that is, for  $n = 1$ . Then we will see how the general case follows from this special case.

When there is one interval, the error estimate becomes

$$\left| \frac{f(a) + f(b)}{2}(b-a) - \int_a^b f(x) dx \right| \leq \frac{K_2(b-a)^3}{12},$$

which is equivalent to

$$(2) \quad -\frac{K_2(b-a)^3}{12} \leq \frac{f(a) + f(b)}{2}(b-a) - \int_a^b f(x) dx \leq \frac{K_2(b-a)^3}{12}.$$

To prove this, we introduce a new function  $E: [a, b] \rightarrow \mathbb{R}$  defined as

$$E(x) = \frac{f(a) + f(x)}{2}(x-a) - \int_a^x f(t) dt.$$

Note that  $E(x)$  is the difference between the integral  $\int_a^x f(t) dt$  and its Trapezoid sum approximation with one interval. Thus  $E(x)$  is the error made by this approximation.

We have

$$E'(x) = \frac{f'(x)}{2}(x-a) + \frac{f(a) + f(x)}{2} - f(x) = \frac{f'(x)}{2}(x-a) + \frac{f(a) - f(x)}{2}$$

and

$$E''(x) = \frac{f''(x)}{2}(x-a) + \frac{f'(x)}{2} - \frac{f'(x)}{2} = \frac{f''(x)}{2}(x-a).$$

The assumption about  $K_2$  is equivalent to  $-K_2 \leq f''(x) \leq K_2$  for all  $x$  in  $[a, b]$ . It follows that

$$-\frac{K_2}{2}(x-a) \leq E''(x) \leq \frac{K_2}{2}(x-a)$$

for all  $x$  in  $[a, b]$ . Switching from  $x$  to  $t$  we have

$$-\frac{K_2}{2}(t-a) \leq E''(t) \leq \frac{K_2}{2}(t-a).$$

Integrating from  $a$  to  $x$ , the monotonicity property of integrals implies

$$-\int_a^x \frac{K_2}{2}(t-a) dt \leq \int_a^x E''(t) dt \leq \int_a^x \frac{K_2}{2}(t-a) dt,$$

which simplifies to

$$-\frac{K_2}{4}(x-a)^2 \leq E'(x) - E'(a) \leq \frac{K_2}{4}(x-a)^2.$$

From the formula for  $E'$  we see that  $E'(a) = 0$ . Thus

$$-\frac{K_2}{4}K_2(x-a)^2 \leq E'(x) \leq \frac{K_2}{4}(x-a)^2$$

for all  $x$  in  $[a, b]$ . Applying the monotonicity property again, this time integrating from  $a$  to  $b$ , we have

$$-\int_a^b \frac{K_2}{4}K_2(x-a)^2 dx \leq \int_a^b E'(x) dx \leq \int_a^b \frac{K_2}{4}(x-a)^2 dx,$$

which simplifies to

$$-\frac{K_2}{12}(b-a)^3 \leq E(b) - E(a) \leq \frac{K_2}{12}(b-a)^3.$$

From the formula for  $E$  we see that  $E(a) = 0$ . Thus

$$-\frac{K_2}{12}(b-a)^3 \leq E(b) \leq \frac{K_2}{12}(b-a)^3,$$

which is what we wanted to prove in (2).

To see that the general case follows from the case for one interval, we restate (2) for the subinterval  $[x_{k-1}, x_k]$ :

$$-\frac{K_2(\Delta x)^3}{12} \leq \frac{f(x_{k-1}) + f(x_k)}{2}\Delta x - \int_{x_{k-1}}^{x_k} f(x) dx \leq \frac{K_2(\Delta x)^3}{12}.$$

Since  $\Delta x = (b-a)/n$ , this becomes

$$-\frac{K_2(b-a)^3}{12n^3} \leq \frac{f(x_{k-1}) + f(x_k)}{2}\Delta x - \int_{x_{k-1}}^{x_k} f(x) dx \leq \frac{K_2(b-a)^3}{12n^3}.$$

Summing the three parts of this inequality as  $k$  goes from 1 to  $n$  we obtain

$$-\frac{K_2(b-a)^3}{12n^2} \leq T_n - \int_a^b f(x) dx \leq \frac{K_2(b-a)^3}{12n^2},$$

which is equivalent to (1).  $\square$

**Extra Credit Problem.** Modify this to get a proof for the midpoint sum error estimate.

**Monotonicity and the Speed Limit Law.** If you took Math 111 using the text by Oseebe & Zorn, you saw a property of derivatives they call the Speed Limit Law: If  $F: [a, b] \rightarrow \mathbb{R}$  is differentiable and  $F'(x) \leq K$  for all  $x$  in  $[a, b]$ , then  $F(b) - F(a) \leq K(b - a)$ . If  $F'$  is non-negative and we think of  $x$  as time and  $F(x)$  as the position of a particle at time  $x$ , this says that if the speed of the particle is no greater than  $K$ , then the distance  $F(b) - F(a)$  the particle goes during the time interval  $[a, b]$  is no greater than  $K(b - a)$ . More briefly (but less precise), the distance is no greater than the maximum speed times time.

We can abbreviate the Speed Limit Law as

$$(3) \quad F'(x) \leq K \text{ for } x \text{ in } [a, b] \text{ implies } F(b) - F(a) \leq K(b - a).$$

Suppose  $f: [a, b] \rightarrow \mathbb{R}$  is integrable and  $f(x) \leq K$  for all  $x$  in  $[a, b]$ . The monotonicity property of integrals implies  $\int_a^b f(x) dx \leq \int_a^b K dx = K(b - a)$ . We can abbreviate this special case of monotonicity as

$$(4) \quad f(x) \leq K \text{ for } x \text{ in } [a, b] \text{ implies } \int_a^b f(x) dx \leq K(b - a).$$

These two properties, one for derivatives and one for integrals, are very intuitive, and for different reasons. They are, of course, related by the Fundamental Theorem of Calculus: if  $f$  is integrable and  $F' = f$ , then the two statements say the same thing.

If you take Math 333, you will see proofs of (3) and (4). Interestingly, they have separate proofs and mathematicians think of them as different theorems, even though it seems like they are the same just by letting  $f$  be  $F'$ . While these properties are equivalent for functions you are used to (differentiable functions with continuous derivatives), they are not equivalent in general. It turns out that there are differentiable functions  $F$  for which  $f = F'$  is discontinuous, and so horribly discontinuous that  $f$  is not integrable! Such a function is way beyond the scope of this handout, but can be found on page 107, item 35, of the book *Counterexamples in Analysis*.

#### REFERENCE

Gelbaum & Olmstead, *Counterexamples in Analysis*, Holden-Day, San Francisco, 1964.

ROBERT FOOTE, OCTOBER 1999, REVISED DECEMBER 2013