PROOF OF THE TRAPEZOID SUM ERROR ESTIMATE

Math 112

Theorem. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is twice-differentiable. Let*

$$T_n \;=\; \sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2}\Delta x \;=\; \left(\frac{f(x_0)}{2} + \sum_{k=1}^{n-1} f(x_k) + \frac{f(x_n)}{2}\right)\Delta x,$$

where n is a positive integer, $\Delta x = (b-a)/n$, and $x_k = a + k\Delta x$ for $k = 0, 1, \ldots n$. Then

$$\left| T_n - \int_a^b f(x)\,dx \right| \le \frac{K_2(b-a)^3}{12n^2}, \tag{1}$$

where K_2 is any number such that $|f''(x)| \le K_2$ for all x with $a \le x \le b$.

The proof is based on an application of the "Racetrack Principle," which is a generalization of the Speed Limit Law you studied and used if you took Math 111. They both express reasonable ways functions and their derivatives should be related. The Racetrack Principle is discussed and proved in Section 4.9 of Ostebee & Zorn (our calculus text) as a consequence of the Mean Value Theorem.

Racetrack Principle. *Suppose that $f, g\colon [a,b] \to \mathbb{R}$ are differentiable, and that $f'(x) \le g'(x)$ for all x in $[a,b]$.*

(1) *If $f(a) \le g(a)$, then $f(x) \le g(x)$ for all x in $[a,b]$.*
(2) *If $f(b) \ge g(b)$, then $f(x) \ge g(x)$ for all x in $[a,b]$.*
(3) *If $a \le x_1 \le x_2 \le b$, then $f(x_2) - f(x_1) \le g(x_2) - g(x_1)$.*

Proof of the error estimate. First we will prove (1) for one interval, that is, for $n = 1$. Then we will see how the general case follows from this special case.

When there is one interval, the error estimate becomes

$$\left| \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)\,dx \right| \le \frac{K_2(b-a)^3}{12},$$

which is equivalent to

$$-\frac{K_2(b-a)^3}{12} \;\le\; \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)\,dx \;\le\; \frac{K_2(b-a)^3}{12}. \tag{2}$$

To prove this, we introduce two new functions $F, G\colon [a,b] \to \mathbb{R}$ defined as

$$F(x) = \frac{f(a)+f(x)}{2}(x-a) - \int_a^x f(t)\,dt \qquad \text{and} \qquad G(x) = \frac{K_2(x-a)^3}{12}.$$

In terms of F and G, what we want to prove is $-G(b) \le F(b) \le G(b)$. To do this we will show by a direct computation that F'' and G'' satisfy an inequality. By two successive applications of the Racetrack Principle we will get inequalities between F' and G', and then between F and G.

We have

$$F'(x) \;=\; \frac{f'(x)}{2}(x-a) + \frac{f(a)+f(x)}{2} - f(x) \;=\; \frac{f'(x)}{2}(x-a) + \frac{f(a)-f(x)}{2},$$

$$F''(x) \;=\; \frac{f''(x)}{2}(x-a) + \frac{f'(x)}{2} - \frac{f'(x)}{2} \;=\; \frac{f''(x)}{2}(x-a),$$

$$G'(x) = \frac{K_2(x-a)^2}{4}, \qquad \text{and} \qquad G''(x) = \frac{K_2(x-a)}{2}.$$

The assumption about K_2 is equivalent to $-K_2 \le f''(x) \le K_2$ for all x in $[a,b]$. It follows that $-G''(x) \le F''(x) \le G''(x)$ for all x in $[a,b]$. We also have $F'(a) = G'(a) = 0$. Applying part (1) of the Racetrack Principle to F' and G' we have $-G'(x) \le F'(x) \le G'(x)$ for all x in $[a,b]$. Noting that $F(a) = G(a) = 0$, we can apply the Racetrack Principle again, this time to F and G, obtaining $-G(x) \le F(x) \le G(x)$ for all x in $[a,b]$. In particular, this inequality holds when $x = b$, which is what we wanted to prove.

To see that the general case follows from the case for one interval, we restate (2) for the subinterval $[x_{k-1}, x_k]$:

$$-\frac{K_2(\Delta x)^3}{12} \;\le\; \frac{f(x_{k-1})+f(x_k)}{2}\Delta x - \int_{x_{k-1}}^{x_k} f(x)\,dx \;\le\; \frac{K_2(\Delta x)^3}{12}.$$

Since $\Delta x = (b-a)/n$, this becomes

$$-\frac{K_2(b-a)^3}{12n^3} \;\le\; \frac{f(x_{k-1})+f(x_k)}{2}\Delta x - \int_{x_{k-1}}^{x_k} f(x)\,dx \;\le\; \frac{K_2(b-a)^3}{12n^3}.$$

Summing the three parts of this inequality as k goes from 1 to n we obtain

$$-\frac{K_2(b-a)^3}{12n^2} \;\le\; T_n - \int_a^b f(x)\,dx \;\le\; \frac{K_2(b-a)^3}{12n^2},$$

which is equivalent to (1).

Extra Credit Problem. Modify this to get a proof for the midpoint sum error estimate.

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