SERIES REVIEW SHEET, SECTIONS 11.1 TO 11.4

Fill in the blanks and give the indicated examples, **including reasons**. Turn in a copy and keep a copy for yourself.

Don't simply fill in the blanks and give the examples. Take this opportunity to really review this material. The credit you get for simply filling in the blanks is nearly insignificant compared to the credit you will get on the BSE and the next exam by really knowing this.

Geometric Series.

- A series is geometric if there is a number r such that ______________________________.
- A geometric series converges if ______________.
- A geometric series diverges if ______________.
- When a geometric series converges, it converges to (or its sum is) ______________, where a is ______________________________.
- Give two examples of convergent geometric series and their sums, one with positive ratio and one with negative ratio.

- Give an example of a divergent geometric series.

p-Series. A p-series is any multiple of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ or its tail, where p is a positive constant.

- A p-series converges if ______________.
- A p-series diverges if ______________.
- The series that divides convergent and divergent p-series is ______________.
- Give three examples of p-series, one that converges and two that diverge.

Notation. The notation $\sum_n^\infty a_n$ indicates a series in which the index n starts at some value. For purposes of convergence or divergence, it doesn't matter where the index starts.

n^{th} **Term Test.** If $\lim_{n\to\infty} a_n$ is not zero, then $\sum_n^\infty a_n$ ____________________________.

- Note: This test cannot be used to conclude that a series converges, that is, if $\lim_{n\to\infty} a_n = 0$ the series does not have to converge.
- Give three examples of divergent series with terms that go to 0.

Comparison Test. Suppose $0 \leq a_n \leq b_n$ for all sufficiently large n.

- If $\sum_n^\infty b_n$ converges, then ______________________________.
- If $\sum_n^\infty a_n$ diverges, then ______________________________.
- If $\sum_n^\infty a_n$ converges, then ______________________________.
- If $\sum_n^\infty b_n$ diverges, then ______________________________.
- Geometric and p-series are good to use for comparisons.
- Error estimates: If $R_n = \displaystyle\sum_{k=n+1}^{\infty} a_k$ is the remainder for the smaller series and $\tilde{R}_n = \displaystyle\sum_{k=n+1}^{\infty} b_k$ is the remainder for the larger series, then $R_n < \tilde{R}_n$. This can be useful in determining a bound on the error, R_n, for the smaller series if a bound on the error of the larger series, $\tilde{R}_n$, can be obtained. The two main cases are when the larger series is geometric or is a p-series. If $\sum_n^\infty b_n$ is geometric, you have an exact formula for $\tilde{R}_n$. If $\sum_n^\infty b_n$ is a p-series, then $\tilde{R}_n$ can be estimated by an easy improper integral.
- Give two examples of the use of the comparison test, one for a convergent series and one for a divergent series. Give complete details.

Integral Test. Suppose $a(x)$ is a positive, continuous, decreasing function and that $a_n = a(n)$ when n is a positive integer. Suppose that $n = N$ is the first index for the series.

- If $\int_N^\infty a(x)\,dx$ converges, then $\sum_n^\infty a_n$ ____________________________.
- If $\int_N^\infty a(x)\,dx$ diverges, then $\sum_n^\infty a_n$ ____________________________.
- Error estimate for the tail: $R_n <$ ____________________________.

The Comparison and Integral Tests may be used only on series of positive terms. (If you have a series of all negative terms, you can use these tests on the negative of the series.)

Improved Ratio Test. Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

- If $L < 1$, then $\sum_n^\infty a_n$ ____________________________.
- If $L > 1$, then $\sum_n^\infty a_n$ ____________________________.
- If $L = 1$, then ____________________________.
- The Improved Ratio Test can be used on any series, regardless of the signs of the terms.

To illustrate how the test can fail, give two example series, one that converges and one that diverges. *For both series you should have $L = 1$.* Give the details.

From your examples, you can probably see that the ratio test should not be used for p-series, or series closely related to p-series.

Alternating Series Test. Consider $\sum_n^\infty (-1)^n c_n$, in which each c_n is positive.

- If (1) $c_1 \geq c_2 \geq c_3 \geq \ldots$ and (2) $\lim_{n\to\infty} c_n = 0$, then $\sum_n^\infty (-1)^n c_n$ __________________________.
- This test cannot be used to conclude that a series diverges.
- In particular, if $\lim_{n\to\infty} c_n$ is not 0, the series diverges, but it's not because of the AST. It diverges because of __________________________.
- Conditions (1) and (2) are independent, that is, it's possible for one of them to be true and the other false. There are series in which (1) is false and (2) is true. Some of them converge and some of them diverge. They have to be analyzed individually.
- Error estimate: If $\sum_n^\infty (-1)^n c_n$ satisfies the AST and R_n is the remainder, then $|R_n| <$ ______________. Since this is so easy, this is a very nice error estimate when it applies.
- Give an example of a divergent alternating series in which the terms go to zero. Explain.

Absolute and Conditional Convergence.

- $\sum_n^\infty |a_n|$ is what you get when you change all of the negative terms to positive (that is, you replace each negative a_n with $|a_n|$, which is positive).
- If $\sum_n^\infty |a_n|$ converges, so does ____________, in which case we say that ____________converges absolutely.
- It's possible for $\sum_n^\infty a_n$ to converge and for $\sum_n^\infty |a_n|$ to diverge. In this case we say that $\sum_n^\infty a_n$ converges _________________________. Give two examples of series that do this with complete details, *other than the alternating harmonic series.*

- Error estimates: If $\sum_n^\infty a_n$ converges absolutely, then a remainder for $\sum_n^\infty a_n$ is bounded by the corresponding remainder for $\sum_n^\infty |a_n|$. More precisely, if $R_n = \sum_{k=n+1}^\infty a_k$ and $\tilde{R}_n = \sum_{k=n+1}^\infty |a_k|$, then $|R_n| < \tilde{R}_n$.
- If $\sum_n^\infty a_n$ is a convergent series of positive terms, then it automatically converges absolutely, since it's equal to $\sum_n^\infty |a_n|$.
- The categories of absolute convergence, conditional convergence, and divergence cover all series and they are non-overlapping: every series is in exactly one category. There are no notions of conditional or absolute divergence.