# Common Errors on Infinite Series Problems

Showing that a series converges absolutely, converges conditionally, or diverges involves a logical argument, and not just computations. For a complete argument, you must give a reason for each series or improper integral you claim converges or diverges. Some can be done directly, but most need one of the convergence tests. When you use a convergence test, you need to verify its conditions before making a conclusion from it.

## Language and Logical Pitfalls.

1. Error: In a problem involving more than one series, or a series and an integral, not being clear which you are refering to.

2. Error: Using the word "it" is often ambiguous. In general, avoid using "it" for a mathematical object, unless there is only one in the discussion (which is rare).

3. Solution: It helps to write in complete sentences, or at least clear sentence fragments. I'm not going to grade on how good your writing is, but your thought process needs to be clear to me, and much of the logical argument is necessarily in words.

## Misuse, confusing, or conflicting use of notation.

This often arises when confusing a series and a sequence, or a series and an integral, or an integral and the function being integrated.

1. Error: Having $\sum_{n=1}^{\infty}$ when you shouldn't, or not having $\sum_{n=1}^{\infty}$ when it is needed.

2. Error: Calling $\sum_{n=1}^{\infty} \frac{n}{n^3+4}$ a $p$-series. Solution: It is closely related to the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

3. Error: Calling $1/x^2$ a $p$-series or improper $p$-integral — it is a function.

4. Error: Calling $\int_1^{\infty} \frac{1}{x^2}\,dx$ a $p$-series — it is an improper $p$-integral.

5. Error: Treating $\int_1^{\infty} \frac{1}{x^2}\,dx$ as a function. It doesn't make sense to let $f(x) = \int_1^{\infty} \frac{1}{x^2}\,dx$. Solution: Let $f(x) = 1/x^2$.

6. Error: Saying that $1/x^2$ converges. This is an error because no limit is involved. The limit $\lim_{x\to\infty} 1/x^2$ converges. The limit $\lim_{x\to 0} 1/x^2$ diverges. Convergence and divergence are just alternate ways to say the limit exists or does not exist.

7. When it is understood that $n$ ranges over the positive integers, then it does make sense to say that $1/n^2$ converges because you can interpret it as a sequence. However, it is better to represent the sequence as $1/1^2$, $1/2^2$, $1/3^2$, $\ldots$, or as $(1/n^2)_{n=1}^{\infty}$ to make it clear we are talking about a sequence, or to use the limit notation, $\lim_{n\to\infty} 1/n^2$.

8. Error: Saying that $\sum_{n=1}^{\infty} \frac{n^2}{n^3+8}$ and $\int_1^{\infty} \frac{x^2}{x^3+8}\,dx$ are equal. They are closely related, but not equal.

9. Error: Saying that $\sum_{n=1}^{\infty} \frac{n^2}{n^3+8}$ and $\lim_{n\to\infty} \frac{n^2}{n^3+8}$ are equal. Again, they are closely related, but not equal.

10. You can't just look at the function you are integrating to determine if an improper integral converges or diverges, you also need to consider the endpoints of integration: $\int_1^\infty \frac{1}{x^2}\,dx$ converges, but $\int_0^1 \frac{1}{x^2}\,dx$ and $\int_0^\infty \frac{1}{x^2}\,dx$ both diverge.

11. Misuse of the word "harmonic." There is only one harmonic series, namely $\sum_{n=1}^\infty 1/n$ (although you can start the sum at a number other than 1). There are two alternating harmonic series, $\sum_{n=1}^\infty (-1)^n 1/n$ and $\sum_{n=1}^\infty (-1)^{n+1} 1/n$, but they are just negatives of each other, so the distinction is somewhat superficial.

## Errors with Inequalities

1. Error: The expression $\sum_{n=1}^\infty \frac{n}{n^3+1} < \frac{1}{n^2}$ doesn't make sense because the first part is a series and the second a term of a sequence. Solution: It should be either $\sum_{n=1}^\infty \frac{n}{n^3+1} < \sum_{n=1}^\infty \frac{1}{n^2}$ (both being summed) or $\frac{n}{n^3+1} < \frac{1}{n^2}$ (neither being summed).

2. Error: Stating an inequality without justification. Solution: Do the algebra or look for a different way to do the problem. Note: We have often made a fraction bigger by making its denominator smaller or its numerator bigger. We have done this enough that this does not need to be justified. Here are some examples.

   (a) Error: Saying $\frac{\sqrt{n+2}}{2n^2+n+1} < \frac{1}{n^{3/2}}$ without justification. This one is likely true (and so useful), but it is not obvious and needs justification.

   (b) Error: Saying $\frac{\sqrt{n+2}}{2n^2+n+1} < \frac{\sqrt{n}}{2n^2} = \frac{1}{2n^{3/2}}$ without justification. (In fact this one is not true.)

   (c) In both of these examples, in going from $\frac{\sqrt{n+2}}{2n^2+n+1}$ to $\frac{\sqrt{n}}{n^2}$ or to $\frac{\sqrt{n}}{2n^2}$, the denominator has gotten smaller (which make the fraction bigger), but the numerator has also gotten smaller (which makes the fraction smaller). It takes more work to determine if the fraction has gotten bigger.

   (d) Saying $\frac{\sqrt{n+2}}{2n^2+n+1} < \frac{\sqrt{n+2}}{2n^2} < \frac{\sqrt{n+2}}{n^2}$ is okay because in each step the numerator stays the same and the denominator gets smaller. However, these inequalities are not useful in the Comparison Test.

   (e) Here is a clever way to get this into something more useful: $\frac{\sqrt{n+2}}{2n^2+n+1} < \frac{\sqrt{n+2}}{2n^2} < \frac{\sqrt{n+2n}}{2n^2} = \frac{\sqrt{3}\sqrt{n}}{2n^2} = \frac{\sqrt{3}}{2}\frac{1}{n^{3/2}}$.

## Misuse of Convergence Tests

1. Error: Citing a test without checking its details.

2. Error: Not saying what test you are using.

3. Error: Citing a non-existent test! (It happens a lot!)

4. Error: Using the formula $\frac{a}{1-r}$ to determine convergence or divergence of a geometric series. Solution: You can use this formula only after you determine the series converges. Use the value of $r$ to determine convergence or divergence.

5. Error: Citing a test that you haven't used or that doesn't apply to the series you are checking. Solution: Use the integral and comparison tests only on series of positive terms.

6. Error: Using the integral or one of the comparison tests on an alternating series. Solution: It may be appropriate to use these on the related series of positive terms.

7. Error: Saying that $1/\sqrt{n}$ diverges by the $p$-test because $p = 1/2$ (it's not a series). Solution: the series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges by the $p$-test because $p = 1/2$. In fact, interpreted as a *sequence*, $1/\sqrt{n}$ converges because the limit $\lim_{n\to\infty} 1/\sqrt{n}$ exists.

8. Error: Making an invalid conclusion from the Comparison Test.
   - If the larger series diverges, you cannot conclude anything about the smaller series.
   - If the smaller series converges, you cannot conclude anything about the larger series.

9. For the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$, noting that $\lim_{n\to\infty} \frac{\sqrt{n+2}}{2n^2+n+1} = 0$ is true, but not useful. In particular, it is not a valid use of the Limit Comparison Test.

10. Error: Using the Ratio or Root Test on a series that is closely related to a $p$-series. The limit is always 1, and both tests fail in this case.

11. Error: Using the Ratio or Root Test at the endpoints of the interval of convergence of a power series. This is precisely when these tests fail.

12. Error: Forgetting to use absolute value with the Ratio or Root Test.

13. Error: Using the Alternating Series Test (AST) to conclude that a series diverges. Solution: If the terms don't go to zero, use the Divergence Test. If $b_n \geq b_{n+1}$ holds only part of the time, then the series might still converge.

14. Error using the AST: Checking $b_n \geq b_{n+1}$ only for one or two terms. It's okay to write $1/1 > 1/2 > 1/3 > \ldots$ when the pattern is clear. The dots indicate that you have at least thought about the whole sequence of terms. It's not safe to do this if the pattern isn't immediately clear. Solution: Let $f(x)$ be the function that gives the terms of the series and show that $f'(x)$ is eventually negative; Show that $b_n/b_{n+1} \geq 1$ or $b_n - b_{n+1} \geq 0$.

15. Error using the AST: Writing down $b_n \geq b_{n+1}$ and $\lim_{n\to\infty} b_n$ without substituting in the terms of the particular series you are testing. I need to know that you know what to plug in!

16. Error: Using the AST when the series converges absolutely. This is not wrong as much as it is unhelpful.

## Misc.

1. Errors: Not recognizing a geometric series. Saying a series is geometric when it isn't.

2. Error: Misidentifying the value of $p$ for a series that is closely related to a $p$-series: $\sum_{n=1}^{\infty} \frac{n}{n^3+4}$ is closely related to the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, but not to the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

3. Be aware of series with "hidden" negative terms. It is an error to use the Comparison Test directly on $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ because $\sin 4n$ is positive sometimes and negative sometimes. The AST does not apply because the terms switch from postive to negative in an irregular way. Solution: Consider $\sum_{n=1}^{\infty} \frac{|\sin 4n|}{4^n}$. You can use the Comparison Test on this.

You may think I am being overly picky, but mathematics only works if you are very clear and precise about what you mean, and this requires accuracy in saying it. Math 112 is a good place to begin paying attention to this.