CONVERGENCE TESTS FOR SERIES: COMMENTS AND PROOFS PART I: THE COMPARISON AND INTEGRAL TESTS

Math 112

The convergence tests for series have nice intuitive reasons why they work, and these are fairly easy to turn into rigorous proofs. In these notes we will prove the standard convergence tests and give two tests that aren't in our text.

It's important to remember that the convergence or divergence of a series depends only on what happens to some tail of the series—the inclusion or omission of a finite number of terms cannot change a convergent series into a divergent one or vice versa. (If the series converges, the sum is affected, of course.) Because of this, we can let a little convenient sloppiness into our notation. When it doesn't matter what the starting point is for a series, we can write $\sum a_k$ instead of $\sum_{k=1}^{\infty} a_k$ (or $\sum_{k=5}^{\infty} a_k$ or $\sum_{k=k_0}^{\infty} a_k$). More specifically, the notation $\sum a_k$ will always mean an infinite sum, and will only be used when the starting point for the sum is not important.

Slight Generalizations of the Comparison and Integral Tests

The observation above allows the use of the comparison test in cases where the terms of the two series satisfy the appropriate inequality after some point, but perhaps not for the first several terms. Here is the slightly more general comparison test.

Theorem (Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series. Suppose there is an index K such that $0 \leq a_k \leq b_k$ for all $k \geq K$. If $\sum b_k$ converges, so does $\sum a_k$. Similarly, if $\sum a_k$ diverges, so does $\sum b_k$, and both sums are infinite.*

The integral test has a similar modification.

Theorem (Integral Test). *Let $\sum a_k$ be a series. Suppose that K is a positive integer and that $a\colon [K,\infty) \to \mathbb{R}$ is a continuous, decreasing function such that $a_k = a(k)$ for each integer $k \geq K$. Then either the series $\sum a_k$ and the improper integral $\int_K^\infty a(x)\,dx$ both converge, or they both diverge.*

The point here is that the terms of the series and the function don't have to be related for x and k less than K. In fact the function doesn't have to be continuous, decreasing, or even defined. Note that for $K > 1$, if the integral $\int_K^\infty a(x)\,dx$ converges, we can conclude that the series $\sum_{k=1}^\infty a_k$ converges (assuming the index for the series begins with 1). However, if the series converges, we cannot conclude that the integral $\int_1^\infty a(x)\,dx$ converges without additional information about the function a.

The Comparison Test. The comparison test is pretty intuitive. If $a_k \leq b_k$ for all $k \geq K$, then it should be the case that $\sum_{k=K}^\infty a_k \leq \sum_{k=K}^\infty b_k$. If $\sum_{k=K}^\infty b_k$ is finite, so is $\sum_{k=K}^\infty a_k$. Similarly, if $\sum_{k=K}^\infty a_k$ is infinite, so is $\sum_{k=K}^\infty b_k$. It's not hard to make this rigorous.

Proof of the Comparison Test. For $n \geq K$, we have the inequality $\sum_{k=K}^n a_k \leq \sum_{k=K}^n b_k$ between the partial sums. Taking the limit as $n \to \infty$ we have $\lim\limits_{n\to\infty} \sum_{k=K}^n a_k \leq \lim\limits_{n\to\infty} \sum_{k=K}^n b_k$ (which is the same as $\sum_{k=K}^\infty a_k \leq \sum_{k=K}^\infty b_k$) *provided that the limits exist* (the limit ∞ is allowed). These limits exist due to the following lemma about increasing sequences.

Lemma 1. *Suppose c_n, $n = 1, 2, 3, \ldots$, is an increasing sequence. If there is some number L such that $c_n \leq L$ for all n, then $\lim\limits_{n\to\infty} c_n$ exists as a finite number. Otherwise $\lim\limits_{n\to\infty} c_n = \infty$.*

There are two ways to think about this informally.

(1) If an increasing sequence is bounded above, it has to converge (meaning that the limit is finite).

(2) An increasing sequence has to have a limit (which may be infinite).

In the proof of the comparison this lemma is applied to the partial sums $A_n = \sum_{k=K}^n a_k$ and $B_n = \sum_{k=K}^n b_k$, which are increasing since the terms of these sums are non-negative. By the lemma, the limits $\lim\limits_{n\to\infty} A_n = \sum_{k=K}^\infty a_k$ and $\lim\limits_{n\to\infty} B_n = \sum_{k=K}^\infty b_k$ exist (they may be infinite), and then the reasoning preceding the lemma may be applied.

The lemma is pretty intuitive also, but its proof uses one of the fundamental properties of the real numbers, and so it's worth looking at. (You can skip the rest of this section if you're not interested.)

Proof of the lemma. Let's take care of the easy part first. Suppose that the sequence is not bounded above. This means that for every number L, no matter how large, some term of the sequence is larger than L, that is, there is some index N such that $c_N > L$. Since the sequence is increasing, we have $c_n \geq c_N > L$ for all $n \geq N$. Thus, for every number L, there is an index N such that $c_n > L$ for all $n \geq N$. This is exactly what $\lim_{n\to\infty} c_n = \infty$ means.

Now suppose that $c_n \leq L$ for all n, that is L is an upper bound for the sequence. Now L is not the only upper bound—any number greater than L will also be an upper bound, and some numbers less than L may also be upper bounds. Of these upper bounds, one of them will be the smallest. Let ℓ be the least upper bound. I claim that ℓ is the limit of the sequence. To see this, suppose that $a < \ell$. Since ℓ is the least upper bound of the sequence, then a is not an upper bound. This means that some term of the sequence is bigger than a, that is $c_N > a$ for some index N. Since the sequence is increasing we have that $a < c_N \leq c_n \leq \ell$ for every $n \geq N$. Thus for any $a < \ell$ there is an integer N such that $a < c_n \leq \ell$ for all $n \geq N$. This is (essentially) what $\lim_{n\to\infty} c_n = \ell$ means.

In general, $\lim_{n\to\infty} c_n = \ell$ means that for any open interval containing ℓ, the sequence eventually gets inside the interval, and stays inside. More precisely, if a and b are any numbers such that $a < \ell < b$, then there is an index N such that $a < c_n < b$ for all integers $n \geq N$. (You may have learned this as "for any $\epsilon > 0$ there is an index N such that $\ell - \epsilon < c_n < \ell + \epsilon$, or $|c_n - \ell| < \epsilon$, for all $n \geq N$." This just makes the interval around ℓ symmetric, but that's not important.) In the lemma we only need to deal with numbers less than ℓ, since ℓ is an upper bound for the sequence.

The fundamental property of the real numbers used in the lemma is the *least upper* *bound property.*

Least Upper Bound Property. *A non-empty set of real numbers that is bounded above has a least upper bound.*

This property distinguishes the real numbers from the rational numbers. It is crucial to many theorems in calculus and its generalization, analysis. A study of the least upper bound property and its sometimes subtle uses are part of the content of Math 33 and 34.

Extra Credit Challenge Problems.

(1) Give an example of a non-empty set of rational numbers that is bounded above but does not have a rational least upper bound.

(2) Find some calculus theorems that aren't true if you restrict yourself to rational numbers.

The Integral Test. The integral test is also pretty clear once you see the right pictures (see page 578 of OZ). These suggest that

$$\sum_{k=K+1}^{\infty} a_k \leq \int_K^{\infty} a(x)\,dx \leq \sum_{k=K}^{\infty} a_k.$$

As with the comparison test, if $\sum_{k=K}^{\infty} a_k$ is finite, so is $\int_K^{\infty} a(x)\,dx$. Similarly, if $\int_K^{\infty} a(x)\,dx$ is finite, so is $\sum_{k=K}^{\infty} a_k$. A rigorous proof involves partial sums and integrals over finite intervals.

Proof of the Integral Test. For $K \leq k \leq x \leq k+1$ we have $a_{k+1} \leq a(x) \leq a_k$, since both the function and the sequence are decreasing. Integrating from k to $k+1$ yields $a_{k+1} \leq \int_k^{k+1} a(x)\,dx \leq a_k$. If we let $a'_k = a_{k+1}$ and $b_k = \int_k^{k+1} a(x)\,dx$, we have $a'_k \leq b_k \leq a_k$. Note that $\sum a'_k$ and $\sum a_k$ are really the same series, their indices are just offset by one. It then follows from the regular comparison test that $\sum a_k$ and $\sum b_k$ either both converge or diverge.

Now what is $\sum_{k=K}^{\infty} b_k$? It should be $\int_K^{\infty} a(x)\,dx$. Indeed,

$$\int_K^{\infty} a(x)\,dx = \lim_{n\to\infty} \int_K^{n} a(x)\,dx = \lim_{n\to\infty} \sum_{k=K}^{n-1} \int_k^{k+1} a(x)\,dx = \lim_{n\to\infty} \sum_{k=K}^{n-1} b_k = \sum_{k=K}^{\infty} b_k.$$

There is one subtlety, however: $\int_K^\infty a(x)\,dx$ isn't defined as $\lim_{n\to\infty} B_n$, where $B_n = \int_K^n a(x)\,dx$; it's defined as $\lim_{X\to\infty} B(X)$, where $B(X) = \int_K^X a(x)\,dx$. The difference is that n is an integer whereas X is a real number. These are slightly different types of limits. They are obviously related, however, and in this case they are equal. This follows from two lemmas, the second of which is the analogue for functions of the lemma in the previous section.

Lemma 2. *Suppose that $f\colon [0,\infty) \to \mathbb{R}$ and that $c_n = f(n)$ for all integers $n \geq 0$. If $\lim_{x\to\infty} f(x)$ exists, then so does $\lim_{n\to\infty} c_n$, and they are equal (they may both be infinite).*

Lemma 3. *Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is increasing. If there is some number L such that $f(x) \leq L$ for all x, then $\lim_{x\to\infty} f(x)$ exists as a finite number. Otherwise $\lim_{n\to\infty} f(x) = \infty$.*

Extra Credit Challenge Problems. In problems 5, 6, and 7 explicit formulas are not necessary. Good descriptions are adequate.

(3) The integral test says that $\int_K^\infty a(x)\,dx$ converges if and only if $\sum_{k=K}^\infty a_k$ converges. As with any "if and only if" statement, there are really two implications. Carefully explain how the two lemmas are used in each of these implications. In particular, both lemmas are needed for one direction, but only one is needed for the other.

(4) Give a careful proof of the two lemmas.

(5) Give an example of a sequence a_k, $k = 0, 1, 2, \ldots$, and a continuous function $a\colon [0,\infty) \to \mathbb{R}$ with $a_k = a(k)$ for each integer $k \geq 0$ such that $\lim_{k\to\infty} a_k$ exists, but $\lim_{x\to\infty} a(x)$ does not.

(6) Give an example of a series $\sum_{k=0}^\infty a_k$ of non-negative terms and a continuous function $a\colon [0,\infty) \to \mathbb{R}$ with $a_k = a(k)$ for each integer $k \geq 0$ such that the series converges but the improper integral $\int_0^\infty a(x)\,dx$ does not.

(7) Give an example of a series $\sum_{k=0}^\infty a_k$ of non-negative terms and a continuous function $a\colon [0,\infty) \to \mathbb{R}$ with $a_k = a(k)$ for each integer $k \geq 0$ such that the improper integral $\int_0^\infty a(x)\,dx$ converges but the series does not.

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