THE IMPROVED RATIO TEST

Math 112

Supplement to Section 11.4 of Ostebee & Zorn

The Comparison, Integral, and Ratio Tests are for series with positive terms. They can be used to determine absolute convergence for a general series $\sum_n a_n$, in which some terms are positive and some are negative, by applying them to $\sum_n |a_n|$. For example, if you use the Comparison or Integral Test to prove that $\sum_n |a_n|$ converges, then you know that $\sum_n a_n$ also converges, in fact, it converges absolutely. On the other hand, if you use the Comparison or Integral Test to prove that $\sum_n |a_n|$ diverges, then you still don't know if $\sum_n a_n$ converges or not; you just know that it doesn't converge absolutely.

The Ratio Test, on the other hand, gives a bit more information. It can distinguish between absolute convergence and divergence. I like to call this the Improved Ratio Test. This is the sense in which the Ratio Test is used in Section 11.4 and beyond.

Theorem. *Let $\sum_n a_n$ be a series of non-zero terms. Let $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, provided the limit exists.*

(1) *If $\rho < 1$, then $\sum_n a_n$ converges absolutely.*
(2) *If $\rho > 1$, then $\sum_n a_n$ diverges.*
(3) *If $\rho = 1$, the test is inconclusive.*

Here is the proof. Note that $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. This is the same thing we would let ρ equal if we applied the Ratio Test to $\sum_n |a_n|$.

(1) If $\rho < 1$, then $\sum_n |a_n|$ converges by the regular Ratio Test. Thus, $\sum_n a_n$ converges absolutely.
(2) Suppose $\rho > 1$. The regular Ratio Test only says that $\sum_n |a_n|$ diverges, that is, $\sum_n a_n$ does not converge absolutely. However, we can squeeze a bit more out. If $\rho > 1$, then $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$. This says that if you go sufficiently far out in the series, you have $\frac{|a_{n+1}|}{|a_n|} > 1$, or $|a_{n+1}| > |a_n|$. It follows that the terms are getting bigger in magnitude. In particular, the terms don't go to zero. The series then diverges by the n-th term test.
(3) For $\rho = 1$, the best examples to use are p-series or alternating p-series. They all yield $\rho = 1$, but some converge absolutely, some converge conditionally, and some diverge.

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