INFORMAL DERIVATIONS OF ARC LENGTH FORMULAS AND ARC LENGTH IN POLAR COORDINATES

Math 112, Supplement to Section 7.1 in OZ

Suppose that $y = f(x)$ is a differentiable function defined for $a \leq x \leq b$. The arc length of the graph of f is given by

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

This formula can be found informally by using infinitesimals. If ds is length of an infinitesimal part of the curve, the infinitesimal Pythagorean theorem gives us

$$ds = \sqrt{dx^2 + dy^2}.$$

You get the length of the curve by adding these up all along the curve, that is, by integrating. Doing algebra with the infinitesimals yields the following:

$$\begin{aligned} L &= \int_C ds = \int_C \sqrt{dx^2 + dy^2} = \int_C \sqrt{dx^2 + dy^2}\, \frac{dx}{dx} \\ &= \int_a^b \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + f'(x)^2}\, dx. \end{aligned}$$

Polar Coordinates. Now suppose we have the graph of a function in polar coordinates, that is, $r = r(\theta)$ for $\alpha \leq \theta \leq \beta$. The formula for arc-length in terms of r and θ can be derived by using the formulas that relate polar and Cartesian coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Taking differentials of these yields

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta, \qquad dy = \sin\theta\, dr + r\cos\theta\, d\theta.$$

Now do the algebra. Square these expressions for dx and dy and add the results. We get

$$ds^2 = dx^2 + dy^2 = \cdots = dr^2 + r^2\,d\theta^2.$$

(You should fill in the details.) The length of the curve is then

$$\begin{aligned} L &= \int_C ds = \int_C \sqrt{dr^2 + r^2\,d\theta^2} = \int_C \sqrt{dr^2 + r^2\,d\theta^2}\,\frac{d\theta}{d\theta} \\ &= \int_\alpha^\beta \sqrt{\frac{dr^2}{d\theta^2} + r^2\frac{d\theta^2}{d\theta^2}}\,d\theta = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = \int_\alpha^\beta \sqrt{r'(\theta)^2 + r(\theta)^2}\,d\theta. \end{aligned}$$

Thus, the arc length formula in polar coordinates is

$$L = \int_\alpha^\beta \sqrt{r'(\theta)^2 + r(\theta)^2}\,d\theta.$$

Historical Note. These informal derivations with infinitesimals are fairly close to how these formulas were originally discovered. They are not rigorous, however, because we haven't given a rigorous definition of infinitesimals. They are intuitive, though, which is why it's worth learning and using them.

Exercises

Use the formula for arc length in polar coordinates to find the lengths of the following curves. You need to know what the graph looks like to determine α and β.

You may need *Mathematica* to evaluate some of the integrals, and some of them may need to be approximated numerically. In some cases you may need to use

NIntegrate[**F**[θ], {θ,α,β}] instead of $\int_\alpha^\beta F(\theta)\,d\theta$.

(1) $r = 1 + \sin\theta$. Cardioid.
(2) $r = \sin\theta$. This is a circle, so you can check your answer.
(3) $r = \cos 2\theta$. Four-leaf clover.
(4) $r = \sin 3\theta$. Three-leaf clover.

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