

A GEOMETRIC SOLUTION TO THE CAUCHY PROBLEM FOR THE HOMOGENEOUS MONGE-AMPÈRE EQUATION

ROBERT L. FOOTE

Let $\Omega \subset \mathbb{R}^n$ be open and connected, and let $u: \Omega \rightarrow \mathbb{R}$ be C^2 . The Hessian of u is the symmetric tensor $H(u) = D du$, where D is the standard flat covariant derivative on \mathbb{R}^n . If X and Y are vectors at $p \in \Omega$, then $H(u)(X, Y) = (D_X du)(Y) = X(du(Y)) - du(D_X Y)$, where Y is extended in a smooth way to a neighborhood of p . The homogeneous Monge-Ampère equation is

$$\det H(u) = 0.$$

If (x_1, \dots, x_n) is an affine coordinate system on \mathbb{R}^n , then $H(u) = \sum_{i,j} u_{ij} dx_i \otimes dx_j$, and the Monge-Ampère equation becomes $\det(u_{ij}) = 0$. Indeed, the Hessian and Monge-Ampère equation take this form in an affine coordinate system on an arbitrary affine manifold, that is, a manifold with a covariant derivative for which the torsion and curvature vanish. What follows will make sense in this more general setting, and so we will assume henceforth that Ω is an open and connected subset of an affine manifold of dimension n and that D denotes the covariant derivative. This affine manifold will be referred to, when necessary, as the “ambient space.”¹

In general, any equation involving the determinant of a Hessian is called a Monge-Ampère equation. They arise frequently in differential geometry, particularly in problems of prescribing curvature and in variational problems involving curvature. See the survey article by Yau in [Y], and [K]. The non-homogeneous Monge-Ampère equation $\det H(u) = f \neq 0$ is elliptic or hyperbolic (in the nonlinear sense) depending on n and the sign of f , and has been studied by many people using PDE techniques going back to Pogorelov [P]. The homogeneous equation, on the other hand, is parabolic, and is best studied by geometric techniques. This was done, for example by Hartman and Nirenberg [HN]. For more references, the reader is referred to the bibliographies of [K], and a number of the papers in [Y].

Research partially supported by the Byron K. Trippet Research Fund at Wabash College. Part of the research was conducted while the author was visiting the Pohang Institute of Science and Technology. He would like to thank the members of the mathematics department there for their hospitality during his visit.

¹The problem we study is local (to a hypersurface) in nature, and so the theory is no different than in \mathbb{R}^n . We choose the more general setting, however, to emphasize that our results depend only on the affine structure of the ambient space, and not on the metric structure of \mathbb{R}^n .

The study of Monge-Ampère equations also arises in complex differential geometry. The complex Monge-Ampère equation is $\det(u_{i\bar{j}}) = f$, where u and f are functions on a complex manifold and the derivatives are taken with respect to a holomorphic coordinate system. Much work has been done in this area by Stoll, Bedford, Burns, Taylor, Wong, Patrizio, Lempert, Yau, and others. See the bibliographies of [Y], and [F1] for references.

In this paper we will study the Cauchy problem for the real homogeneous Monge-Ampère equation $\det H(u) = 0$. Let M be an orientable hypersurface of the ambient space, let N be a non-vanishing transverse vector field along M , and let $\varphi, \psi: M \rightarrow \mathbb{R}$. (The exact regularity of M , φ , and ψ will be specified later.) The Cauchy problem we will study concerns existence and uniqueness of functions u defined on a neighborhood of M that satisfy

$$(CP) \quad \begin{cases} \det H(u) = 0 & \text{on a neighborhood of } M, \text{ and} \\ u = \varphi & \text{and } du(N) = \psi \text{ on } M. \end{cases}$$

Solutions of (CP) need not exist, even for C^∞ or C^ω data. In this paper we will investigate generic, geometric conditions that will guarantee the existence and uniqueness of solutions. The essence of our main result is the following:

If $H^\nabla(\varphi) - \psi B$ is non-degenerate on TM , then there exists a unique solution of (CP).

Here $H^\nabla(\varphi)$ denotes the Hessian of φ relative to an induced covariant derivative on M , and B denotes a generalized second fundamental form that depends on M and N . The non-degeneracy condition guarantees that $\text{rank } H(u) = n - 1$, which in turn implies the existence of the so-called Monge-Ampère foliation. The foliation can be constructed from the Cauchy data, resulting in an explicit expression for the solution, which proves uniqueness. Existence follows with the addition of a mild, generic condition, which amounts to a transversality condition on the foliation. We will also see how the regularity of the Cauchy data influences the regularity of the solution. We should note that the Cauchy problem for the complex Monge-Ampère equation was studied by Bedford and Burns in [BB], which motivated the present study.

In the next section we review the relevant geometry of the Monge-Ampère foliation. The Cauchy problem is then addressed in the final section.

THE MONGE-AMPÈRE FOLIATION

Solutions of the homogeneous Monge-Ampère equation $\det H(u) = 0$ for which the Hessian $H(u)$ has constant rank enjoy a geometric property that will be our main tool in solving the Cauchy problem.

THEOREM 1. *Let $u: \Omega \rightarrow \mathbb{R}$ be C^3 with $\text{rank } H(u) = n - k$. Then there is a foliation of Ω by k -planes such that u is affine on each leaf of the foliation.*

This is well-known (see [HN]), but we give a simple, invariant proof for completeness.

PROOF. Let

$$\begin{aligned}\mathcal{F} &= \ker H(u) = \{ X \in T\Omega \mid H(u)(X, Y) = 0 \text{ for all } Y \in T\Omega \} \\ &= \{ X \in T\Omega \mid D_X du = 0 \}.\end{aligned}$$

This is a C^1 distribution in $T\Omega$. Let X and Y be C^1 sections of \mathcal{F} . Let Z be an arbitrary C^1 vector field. Then

$$\begin{aligned}H(u)(Z, D_X Y) &= (D_Z du)(D_X Y) = X((D_Z du)(Y)) - (D_X D_Z du)(Y) \\ &= -(D_Z D_X du)(Y) - (D_{[X, Z]} du)(Y) = 0,\end{aligned}$$

since $X, Y \in \mathcal{F}$ and there is no curvature. Similarly, $H(u)(Z, D_Y X) = 0$. It follows that $D_X Y$, $D_Y X$, and $[X, Y] = D_X Y - D_Y X$ are sections of \mathcal{F} . Thus \mathcal{F} is integrable and the leaves of the resulting foliation are totally geodesic, hence k -planes (see [S], Vol. 3, pg. 32). In directions tangent to the leaves $H(u) \equiv 0$, and so u is affine on each leaf. \square

Clearly the domain of u can be made larger by extending the planes of the foliation beyond Ω and by letting u continue to be affine on the planes. This will be well-defined and will satisfy $\text{rank } H(u) = n - k$ until the planes start to intersect each other. In our application to the Cauchy problem, the rank of $H(u)$ will be $n - 1$, and so the leaves of the foliation will be affine lines. A general study of the geometry of one-dimensional real Monge-Ampère foliations was made by the author in [F2].

It is important to note that having u be affine on each leaf of some foliation of Ω by planes is not a sufficient condition for u to satisfy $\det H(u) = 0$. Indeed, let $f: \mathbb{R} \rightarrow \mathbb{R}$ be C^2 . Then the function $u(x, y) = yf(x)$ is affine on each vertical line of \mathbb{R}^2 and $\det H(u) = -(f'(x))^2$. The proof of the last theorem shows that du must be parallel along the leaves of the foliation, which implies that u is affine along the leaves and that $\ker du$ is parallel along the leaves. As the next theorem shows, these two conditions are generically sufficient to imply that u satisfies $\det H(u) = 0$.

THEOREM 2. *Let $u: \Omega \rightarrow \mathbb{R}$ be C^2 . Suppose that Ω is foliated by k -planes such that u is affine on each plane, $\ker du$ is parallel along each plane, and that the foliation and $\ker du$ are transverse except possibly on a set with no interior. Then du is parallel on each plane. In particular, $\text{rank } H(u) \leq n - k$.*

PROOF. Let X be tangent to a plane of the foliation. If Y is also tangent to a plane of the foliation, then $H(u)(X, Y) = 0$ since u is affine on the plane. If $Y \in \ker du$, then the parallel extension of Y along the plane stays in $\ker du$.

Then $H(u)(X, Y) = X(du(Y)) - du(D_X Y) = 0$. At points where the foliation and $\ker du$ are transverse, we have $H(u)(X, Y) = 0$ for all Y . This holds at points where the transversality condition fails by continuity. \square

The transversality condition is necessary. Consider $u(x, y) = \frac{xy}{x^2+y^2}$ defined on \mathbb{R}^2 except the origin. This is constant on the lines through the origin. Hence it is affine on those lines and $\ker du$ is parallel along those lines. However $\det H(u) = 0$ only on $y = \pm x$.

Before turning to the solution of the Cauchy problem, it is instructive to look at some examples. Let $M \subset \mathbb{R}^n$ be a submanifold (not necessarily a hypersurface) and consider the distance function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $u(x) = \text{dist}(x, M)$. Near M , u is affine on lines orthogonal to M . The vector field $\text{grad } u$ is a unit field pointing "directly away" from M and so is parallel along these lines. This is equivalent to du being parallel along the lines, and so $\text{rank } H(u) \leq n - 1$ (in particular, the kernel of $H(u)$ contains the lines orthogonal to M). Thus u satisfies the homogeneous Monge-Ampère equation on $\Omega \setminus M$, where Ω is some neighborhood of M . The size of Ω and the exact rank of $H(u)$ depend on the extrinsic geometry of M .

Two extreme cases of this are worth noting. First, if M is the origin, then $u(x) = \|x\|$. The leaves of the foliation are the lines through the origin. This is, in some sense, the 'standard solution' of the homogeneous Monge-Ampère equation.

Second, suppose M is an oriented hypersurface and u is the signed distance to M , that is, u is positive on one side of M and negative on the other. Then u is a solution of the Cauchy problem

$$\begin{cases} \det H(u) = 0 & \text{on a neighborhood of } M, \text{ and} \\ u = 0 \text{ and } du(N) = 1 & \text{on } M, \end{cases}$$

where N is the unit normal field along M that gives the orientation.

SOLUTION OF THE CAUCHY PROBLEM

Let M , N , φ , and ψ be as in the statement of the Cauchy problem (CP). For the moment, assume that these are smooth. The exact regularity requirements will be considered in the last subsection. We use the decomposition $T\Omega = TM \oplus \langle N \rangle$ to define a generalized Weingarten map, a generalized second fundamental form, a covariant derivative on TM , and a generalized Hessian of φ . Given $X \in TM$, define WX and $\alpha(X)$ by

$$D_X N = WX + \alpha(X)N,$$

where WX is tangent to M . Then $W: TM \rightarrow TM$ is a tensor and α is a 1-form on M . Given $X \in TM$ and Y a vector field tangent to M , define $\nabla_X Y$ and $B(X, Y)$ by

$$D_X Y = \nabla_X Y + B(X, Y)N,$$

where $\nabla_X Y$ is tangent to M . Define

$$H^\nabla(\varphi)(X, Y) = (\nabla_X d\varphi)(Y) = X(d\varphi(Y)) - d\varphi(\nabla_X Y).$$

It is easily verified that ∇ is a covariant derivative on TM , and that B and $H^\nabla(\varphi)$ are symmetric bilinear forms on TM . Note that if $M \subset \mathbb{R}^n$ and N is a unit normal field along M , then W is the Weingarten map, α vanishes, B is the second fundamental form of M , and $H^\nabla(\varphi)$ is the induced Hessian on M .

Uniqueness and Existence. As noted above, the key to existence and uniqueness of solutions to the Cauchy problem is the non-degeneracy of $H^\nabla(\varphi) - \psi B$.

THEOREM 3. *If $H^\nabla(\varphi) - \psi B$ is non-degenerate on TM , then there exists at most one smooth solution u of (CP).*

PROOF. Suppose the solution of (CP) exists. Then $\text{rank } H(u) \leq n - 1$. Let $X \in TM$ and let Y be a vector field tangent to M . Then

$$\begin{aligned} H(u)(X, N) &= (D_X du)(N) = X(du(N)) - du(D_X N) \\ (1) \quad &= d\psi(X) - du(WX + \alpha(X)N) \\ &= d\psi(X) - d\varphi(WX) - \psi\alpha(X) \end{aligned}$$

and

$$\begin{aligned} H(u)(X, Y) &= (D_X du)(Y) = X(du(Y)) - du(D_X Y) \\ &= X(d\varphi(Y)) - du(\nabla_X Y + B(X, Y)N) \\ &= X(d\varphi(Y)) - d\varphi(\nabla_X Y) - \psi B(X, Y) \\ &= H^\nabla(\varphi)(X, Y) - \psi B(X, Y). \end{aligned}$$

The non-degeneracy of $H^\nabla(\varphi) - \psi B$ implies that $\text{rank } H(u) \geq n - 1$, and so $\text{rank } H(u) = n - 1$, and this must hold on a neighborhood of M . By Theorem 1, this neighborhood admits a Monge-Ampère foliation by lines. The non-degeneracy condition implies that the lines of the foliation are transverse to M , and so at each point of M there is a unique vector $Z \in TM$ such that $Z + N$ is tangent to the foliation. For $X \in TM$ we then have $H(u)(X, Z + N) = 0$, that is,

$$(2) \quad (H^\nabla(\varphi) - \psi B)(X, Z) = d\varphi(WX) + \psi\alpha(X) - d\psi(X).$$

We have $du(Z + N) = d\varphi(Z) + \psi$. Since u is affine along the leaves of the foliation, we get an explicit formula for u . If $p \in M$, then

$$(3) \quad u(p + r(N_p + Z_p)) = \varphi(p) + r(d\varphi_p(Z_p) + \psi(p)),$$

which uniquely determines u in a neighborhood of M . \square

NOTE. In the last equation we denote the exponential map by addition, $\exp_p(rN_p) = p + rN_p$, as we would in \mathbb{R}^n .

The next theorem provides the existence of a solution to the Cauchy problem in a special case. If the leaves of the Monge-Ampère foliation are the lines determined by N , then $d\psi(X) - d\varphi(WX) - \psi\alpha(X) = 0$ by (1). The existence of a solution follows from this last condition, provided an additional assumption is made about ψ , which guarantees the transversality requirement of Theorem 2.

THEOREM 4. *Suppose $d\psi(X) - d\varphi(WX) - \psi\alpha(X) = 0$ for all $X \in TM$, and that the subset of M on which ψ vanishes has no interior. Then $u(p + rN_p) = \varphi(p) + r\psi(p)$ defines a solution to (CP) on a neighborhood of M , and du is parallel along the lines determined by N .*

PROOF. Let Ω be a neighborhood of M of the form $\{p + rN_p \mid p \in M, |r| < \epsilon(p)\}$, where ϵ is some positive, continuous function on M . Clearly u , as defined above, satisfies the Cauchy data and is affine along the lines determined by N . We need to show that $\ker du$ is parallel along these lines.

Extend N to be parallel along the lines determined by N , that is if $p \in M$ and $q = p + rN_p$ for some $r \in \mathbb{R}$, then $N_q = N_p$. (Since Ω is an affine manifold and our arguments take place in a neighborhood of one of the lines, we can freely identify vectors in one tangent space with those in another. Computations can be made as they would be in \mathbb{R}^n .)

Fix $p \in M$, $r \in \mathbb{R}$, and $q = p + rN_p \in \Omega$. We have $du_q(N) = \psi(p)$. Let $X \in T_pM$, and let γ be a curve in M such that $X = \gamma'(0)$. If $\tilde{\gamma}(t) = \gamma(t) + rN_{\gamma(t)}$, then $\tilde{X} = \tilde{\gamma}'(0) = X + r(WX + \alpha(X))N$, thought of as a vector in $T_q\Omega$. Note that as X varies over T_pM , the vector \tilde{X} varies over an $(n-1)$ -dimensional subspace of $T_q\Omega$ that is transverse to N . We have

$$\begin{aligned} du_q(\tilde{X}) &= \left. \frac{d}{dt} \right|_0 u(\gamma(t) + rN_{\gamma(t)}) = \left. \frac{d}{dt} \right|_0 (\varphi(\gamma(t)) + r\psi(\gamma(t))) \\ &= d\varphi_p(X) + rd\psi_p(X). \end{aligned}$$

Let $b \in \mathbb{R}$ and suppose $\tilde{X} + bN \in \ker du_q$. Then

$$(4) \quad d\varphi_p(X) + rd\psi_p(X) + b\psi(p) = 0.$$

Thinking of

$$\tilde{X} + bN = (X + rWX) + (b + r\alpha(X))N$$

as a vector in $T_p\Omega$ and using the hypothesis and (4), we have

$$du_p((X + rWX) + (b + r\alpha(X))N) = d\varphi_p(X + rWX) + (b + r\alpha(X))\psi(p) = 0.$$

It follows that $\ker du_q$ is parallel to $\ker du_p$.

When $\psi(p) \neq 0$, then N_p and $\ker du_p$ are transverse, and so $\det H(u) = 0$ by Theorem 2. \square

Next we consider the existence of a solution u to the Cauchy problem under the more general condition of the non-degeneracy of $H^\nabla(\varphi) - \psi B$. Following the proof of Theorem 3, the leaves of the foliation will be the lines determined by $N + Z$, where Z is the vector field tangent to M that satisfies (2). In this direction the solution will satisfy $du(N + Z) = \psi + d\varphi(Z)$, and so the condition that will imply the transversality requirement is the non-vanishing of $\tilde{\psi} = \psi + d\varphi(Z)$.

THEOREM 5. *Suppose that $H^\nabla(\varphi) - \psi B$ is non-degenerate on TM . Let Z be the vector field defined by (2), and suppose the function $\tilde{\psi} = \psi + d\varphi(Z)$ vanishes on a set without interior. Then (3) defines a solution of the Cauchy problem (CP) on a neighborhood of M , and du is parallel along the lines determined by $N + Z$.*

PROOF. Let $\tilde{N} = N + Z$ and $\tilde{\psi} = \psi + d\varphi(Z)$. It is easily computed that $D_X \tilde{N} = \tilde{W}X + \tilde{\alpha}(X)\tilde{N}$, where

$$\tilde{W}X = WX + \nabla_X Z - (\alpha(X) + B(X, Z))Z \quad \text{and} \quad \tilde{\alpha}(X) = \alpha(X) + B(X, Z).$$

It follows easily that

$$d\tilde{\psi}(X) - d\varphi(\tilde{W}X) - \tilde{\psi}\tilde{\alpha}(X) = 0$$

for all $X \in TM$. The result then follows from the previous theorem. \square

The special case when φ vanishes is worth singling out. The non-degeneracy condition becomes a curvature condition.

COROLLARY. *Suppose that M has non-zero Gaussian curvature (equivalently, the second fundamental form B is non-degenerate) and ψ is non-vanishing. Then there exists a unique solution to the Cauchy problem*

$$\begin{cases} \det H(u) = 0 & \text{on a neighborhood of } M, \text{ and} \\ u = 0 & \text{and } du(N) = \psi & \text{on } M. \end{cases}$$

PROOF. The condition (2) on Z reduces to $\psi B(X, Z) = d\psi(X) - \psi\alpha(X)$, and the result follows from Theorems 3 and 5. \square

REMARK. When $M \subset \mathbb{R}^n$ and N is unit normal along M , we get an interesting explicit formula for Z under the conditions of the previous corollary. In this case $\alpha = 0$, and B and W are related by $B(X, Z) = -\langle X, WZ \rangle$. We have

$$\langle X, WZ \rangle = -(1/\psi)d\psi(X).$$

The curvature condition implies that W is invertible, and so this is equivalent to

$$Z = -(1/\psi)W^{-1}(\text{grad } \psi) = -W^{-1}(\text{grad } \log \psi).$$

Regularity. Finally, we consider the regularity of the solutions. Suppose M and φ are C^k , and N and ψ are C^{k-1} . In order to define W , α , ∇ , B , and $H^\nabla(\varphi)$, we need $k \geq 2$. The proof of the uniqueness theorem, Theorem 3, goes through assuming u is C^k and $k \geq 3$. (The extra degree of differentiability is needed to apply Theorem 1.)

In the first existence theorem, Theorem 4, if $k \geq 2$, then N is at least C^1 , and the implicit function theorem applied to $u(p + rN_p) = \varphi(p) + r\psi(p)$ implies that u is C^{k-1} . In Theorem 5, one degree of differentiability is lost. The quantities $H^\nabla(\varphi)$, B , W , and α are C^{k-2} , and hence so are Z , \tilde{N} , and $\tilde{\psi}$. We need $k \geq 3$ to apply the implicit function theorem to (3) in this case, and we get that u is C^{k-2} . The surprising fact is that we actually get one more degree of regularity for u than is expected in these theorems.

THEOREM 6. Suppose that M and φ are C^k , and N and ψ are C^{k-1} . In Theorem 4, if $k \geq 2$, then u is C^k at points where $\psi \neq 0$. In Theorem 5, if $k \geq 3$, then u is C^{k-1} at points where $\tilde{\psi} \neq 0$.

PROOF. In the proof of Theorem 4, we have that $X + N \in \ker du$ along M , $X \in TM$, if and only if $d\varphi(X) + \psi = 0$. It follows that $\ker du$ is a C^{k-1} distribution along M . Since N is C^{k-1} and $\ker du$ is parallel along the lines determined by N , it follows that $\ker du$ is a C^{k-1} distribution on Ω . Define the projection $P: \Omega \rightarrow M$ by letting $P(q) = p$ if $q = p + rN_p$ for some $r \in \mathbb{R}$. Then P is C^{k-1} . If X is a smooth vector field on Ω , then $X = Y + fN$ is a C^{k-1} decomposition, where $Y \in \ker du$ and $f: \Omega \rightarrow \mathbb{R}$. Then the function $du(X) = f(\psi \circ P)$ is C^{k-1} . It follows that du is C^{k-1} , and so u is C^k .

The statement for Theorem 5 follows by considering \tilde{N} and $\tilde{\psi}$ in place of N and ψ . \square

An important special case is the distance function.

COROLLARY. If $M \subset \mathbb{R}^n$ is C^k , $k \geq 2$, then the signed distance function given by $u(p + rN_p) = r$, where N is a unit normal along M , is C^k in a neighborhood of M .

Clearly u is C^{k-1} . The extra degree of differentiability is surprising. This result also holds for the unsigned distance function on $\Omega \setminus M$ when M is a

submanifold of arbitrary dimension. For direct proofs of this, see [F3] and its references.

References

- [BB] E. Bedford and D. M. Burns, *Holomorphic Mapping of Annuli in \mathbb{C}^n and the Associated Extremal Function*, Annali Della Scuola Normale Superiore Di Pisa, Serie IV **6** (1979), 381–414.
- [F1] R. L. Foote, *Homogeneous Complex Monge-Ampère Equation and Algebraic Embeddings of Parabolic Manifolds*, Indiana U. Math. J. **39** (1990), 1245–1273.
- [F2] R. L. Foote, *Differential Geometry of Real Monge-Ampère Foliations*, Math. Zeit. **194** (1987), 331–350.
- [F3] R. L. Foote, *Regularity of the Distance Function*, Proc. of Amer. Math. Soc. **92** (1984), 153–155.
- [HN] P. Hartman and L. Nirenberg, *On Spherical Image Maps Whose Jacobians do not Change Signs*, Am. J. Math. **81** (1959), 901–920.
- [K] J. L. Kazdan, *Prescribing the Curvature of a Riemannian Manifold*, CBMS Regional Conf. Series, No. 57, Amer. Math. Soc., Providence, Rhode Island, 1985.
- [P] A. V. Pogorelov, *The Regularity of a Convex Surface with Given Gauss Curvature*, Mat. Sbornik **31** (1952), 88–103.
- [S] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, Houston, 1979.
- [Y] S. T. Yau, ed., *Seminar on Differential Geometry*, Annals of Math. Study 102, Princeton Univ. Press, Princeton, New Jersey, 1982.

DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE,
 WABASH COLLEGE CRAWFORDSVILLE, INDIANA 47933, USA
E-mail: footer@wabash.bitnet

DEPARTMENT OF MATHEMATICS, POHANG INSTITUTE OF SCIENCE AND TECHNOLOGY,
 P. O. BOX125 POHANG, 790–330 KOREA