John
Thiry

Problem of the forthight #2

Prove that among the positive numbers $a, 2a, 3a, 4a, \dots, (n-1)a$ $\exists$ at least 1 that differs from an integer by at most $1/n$.

Proof:

Let us divide the distance between two integers into n blocks of width $1/n$.

Case 1: If any of $a, 2a, 3a, \dots, (n-1)a$ lies in the 1^{st} or last block, that would be within $1/n$ of an integer and we would be done

Case 2: Assume none lie within the first or last block. That means each will be within one of the remaing $n-2$ blocks (although perhaps between different pairs of integers).

So, let xa, ya lie in the same block, but perhaps between different integer pairs. (wlog say $x > y$)

Then, $xa - ya$ = some integer + a fractional part. where the fractional part is at most $1/n$ because they are in the same block.

Thus, the number $(x-y)a$, which is among the numbers $a, 2a, 3a, \dots, (n-1)a$, lies at a distance of at most $1/n$ from an integer.